INE PAN Working Paper Series

Paper number 55

# Measuring multidimensional inequality of opportunity

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Warsaw, October 2023

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# Measuring multidimensional inequality of opportunity

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**Abstract**

This paper extends the measurement of inequality of opportunity (IOP) to the case in which individual outcomes are multidimensional (e.g. income, health). We adopt an axiomatic approach to the construction of IOP measures. We characterize two classes of social welfare functions, each endorsing ex ante compensation but different reward principles: (1) utilitarian and (2) inequality averse. For each class we develop implementable conditions analogous to Lorenz dominance and IOP measures. As a major result we axiomatically characterize a multidimensional inequality of opportunity index corresponding to the Tsui (1995) inequality index applied to the framework of equality of opportunity. The measure is sensitive to dependence between outcomes – a distinctive feature of multidimensional distributions, and has a straightforward interpretation. Using the 2019 wave of EU SILC data we show that the multidimensional analysis of inequality of opportunity in Europe reveals important differences with respect to the analysis based on income alone, showing the value of the multidimensional perspective.

**Keywords:** equality of opportunity, multidimensional welfare, multidimensional inequality, Lorenz dominance

**JEL classification:** D63, I32

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# 1 Introduction

Equality of Opportunity (EOP) has in the last decades become an increasingly important goal for public policy. Inclusive growth, with equal opportunities as a core principle, is at the centre of the EU's growth strategy and the European Pillar of Social Rights (European Commission 2021). "Creating opportunities for everyone across the country" (so called 'levelling up) is "a moral, social and economic programme" of the new British government (DLUHC 2022). Equality of opportunity has become a new mantra in politics. Academic research has also flourished. There is by now an extensive empirical literature that studies equal opportunities (see Ferreira and Peragine 2016, Ramos and Van de Gaer 2016, Roemer and Trannoy 2015 for recent surveys) and closely related literature on social mobility (Chetty et al. 2014a, Chetty et al. 2014b). Lack of EOP may be harmful to economic growth (Marrero and Rodriguez 2013, Aiyar and Eveke 2018, Ferreira et al. 2018) thus making EOP a compelling policy goal not only for social justice reasons, but also for purely economic reasons.

EOP is a widely-held ideal of fairness, stating that differences in life success should reflect differences in individual effort and choices, but should not be determined by factors beyond the responsibility of the individual, the so-called circumstances. Building on a distinguished literature in political philosophy (e.g. Rawls 1971, Dworkin 1981, Arneson 1989, Cohen 1989, Roemer 1993, 1998), a vast array of methods have been proposed to quantify the portion of outcome inequality that can be attributed to exogenous circumstances, and is therefore interpreted as inequality of opportunity. The outcome of interest is typically represented by income. On the other hand, since seminal works of Sen (1973), Kolm (1977), and Atkinson and Bourguignon (1982), it has been widely acknowledged that well-being is a multidimensional concept that cannot be reduced to income alone. Increasingly, this stance has also pervaded policy making, as exemplified by the use of the Human Development Index and the OECD's Better Life Index, as well as efforts to measure progress beyond GDP per capita in France, UK and many other countries. Therefore, if equality of opportunity is the goal of public policy, its measurement has to account for the multidimensionality of life outcomes. To date, however, methods to do this have not been developed. We address this problem.

In the basic model of EOP individuals are characterized by two types of variables: factors beyond their responsibility (so called circumstances) and factors within their responsibility (so called effort).[1] One of the most popular ideals of EOP theory is the principle of ex-ante compensation (Van de Gaer 1993) which states that inequalities across people with different circumstances should be equalized before effort is realized. This is implemented by treating outcome distribution among people with the same circumstances – so called "types" – as the opportunity set available to each individual in a type and focusing on inequality between types. The second important ideal of EOP theory is the reward principle, which expresses how to treat inequalities within types, namely, inequalities that arise due to differences in effort that individuals exert. Utilitarian reward principle (Van de Gaer 1993, Fleurbaey 2008) states that they should be neglected, whereas inequality averse reward principle (Ramos and Van de Gaer, 2016) expresses aversion to inequality from effort. In this paper we use ex-ante compensation and two reward principles as defining postulates of equality of opportunity.

We extend the measurement of inequality of opportunity to many dimensions, therefore within each type we have a multidimensional distribution of outcomes. Then, the effect of circumstances may not only be that worse types have worse distributions of outcomes, but also that individuals in worse types are more likely to be deprived in several outcomes than those in better types. The latter concerns dependence, which is a distinctive feature of multidimensionality. The multidimensional measure of inequality of opportunity needs to capture both types of inequality: spread changes in marginal outcome distributions and dependence changes.

As our major result we characterize an inequality of opportunity measure that reflects ex-ante compensation and inequality averse reward (Theorem 4). Both postulates are expressed in the axioms of *inequality aversion between types* and *inequality aversion within types*. These properties together with other properties that are standard for inequality measures, namely, *monotonicity*, *utilitarian aggregation*[2], *ratio scale invariance*, jointly characterize the measure. Character-

[1]There are richer models of EOP that include "luck" component, e.g. see LeFranc et al. (2008).

[2]In a related paper (Kapera and Kobus 2023) we show that this assumption can be relaxed by using a general aggregation function for the aggregation of types. However, then it is often

isation means that the measure not only satisfies these properties, but that it is the only measure that satisfies them, thus making the measurement exercise much less arbitrary. We thus adopt an axiomatic method from the outset, which, unlike the literature on inequality measurement, is not the standard approach in the literature on inequality of opportunity. Here, postulates have not been linked to measures and measures have been proposed in a rather ad hoc way. This has only recently been addressed by Bosmans and Öztürk (2021), who develop a unified framework for unidimensional IOP. We do this from the beginning, which makes the framework and the results more complete and clear.

The measure obtained is very natural as it is a standard multidimensional inequality index (Tsui 1995) applied to the multidimensional inequality of opportunity framework.[3] It allows a simple distinction between inequality between types and inequality within types, which makes it easier to understand. Furthermore, in one dimension it reduces to the Atkinson (1970) index applied to the EOP setting. Finally, as a consequence of the axiomatic method, the measure is related to a welfare function following the approach of Atkinson (1970), Kolm (1969) and Sen (1973), so that it becomes easily interpretable as the proportion of average outcomes lost due to existing inequalities of opportunity.

Let us now briefly describe all the results in more detail. Formally speaking, we have an ordering of the society (multidimensional) outcome distributions, represented by a continuous social evaluation function that reflects the preference relation of a social planner who values equality of opportunity. We begin by characterizing the class of welfare functions that express compensation and reward principles. These axioms are based on transformations that define multidimensional inequality, namely, transformations that reduce spread in many dimensions (i.e., Pigou-Dalton Transfer of the same proportionate amount on each dimension — see e.g. Tsui 1999) and transformations that change the dependence between dimensions (i.e., correlation increasing switches — see e.g. Epstein and Tanny

not possible to derive a concrete inequality index.

[3]The result also embeds a univariate ex ante approach in the multivariate setting; a commonly used univariate measure of inequality of opportunity is the mean logarithmic deviation, which belongs to the family of generalised entropy measures. Tsui's (1995) index is often referred to as multidimensional generalised entropy. Another multivariate extension of generalised entropy comes from Maasoumi (1986). The two classes are related (Aaberge and Brandolini 2015, pp. 195-198).

1980). A welfare function that respects ex-ante compensation changes when such transformations take place *between* types, because then the inequality between types changes. On the other hand, when such transformations take place *within* type, a welfare function that respects utilitarian reward does not change, since it considers all inequality within type irrelevant, and a welfare function that respects inequality averse reward increases.

In Theorem 1 we show that the first class considered (utilitarian reward and ex ante compensation) is implemented via Component-wise Generalized Lorenz Dominance (CGLD) which applies standard Generalized Lorenz Dominance to the vector of type means separately for each outcome. This is a restrictive condition, because it requires that one type is better than another only if it has a higher mean for each outcome. This only reveals the stringency of the utilitarian reward axiom which by neglecting the distribution of outcomes within type actually causes them to be treated separately. The second class (inequality averse reward and ex-ante compensation) does not suffer from such constraints and allows for a truly joint treatment of outcomes. Here the implementable criterion is Generalized Lorenz Dominance applied to the type utilities where utility function must be concave and submodular (Theorem 2). A better type is simply the one that has a higher type utility. Next, using these two classes of welfare functions, inequality of opportunity measures are derived. The IOP measure generated by the first class of welfare functions functions has a very simple form: it corresponds to a weighted sum of the types' means for each attribute, normalized by the value of the highest welfare (Theorem 3). The latter is obtained in the case of perfect equality, both within and between types, i.e. a distribution of population means. The weights assigned to the types preserve the aversion to inequality between types, i.e. the worse type gets a higher weight. The second IOP measure generated by the second class of welfare functions is our main result (Theorem 4) described above.

The paper then provides an empirical application based on the latest wave (2019) of the EU SILC data: we analyse multidimensional inequality of opportunity in 29 European countries, focusing on three outcome variables, namely income, education and health, and six circumstance variables: biological sex, parental education, family composition, place of birth, family financial situation when the respondent was 14 years old, and parental occupation. The results show that the unidimensional inequality of opportunity rankings of countries differ according to the outcome of interest, and also that the impact of circumstances varies by outcome. In particular, countries that provide more equal opportunities for income do not necessarily do so for other other life outcomes. For example, Denmark, Norway and Luxembourg are among the most egalitarian countries in terms of income acquisition, but they are also among the least egalitarian in terms of health and educational opportunities. This is in line with recent research (Carneiro et al. 2021, Heckman and Landersø 2022) which shows that Norway and Denmark achieve social mobility mainly through post-redistribution (i.e. tax and transfer systems) rather than pre-redistribution (i.e. investment in children's human capital), and therefore their educational and health mobility is not as high as income mobility. The multidimensional analysis thus shows that the picture of multidimensional well-being in Europe is different from that suggested by income alone. We also use a K-means algorithm to group European countries according to the combination of the joint IOP and three unidimensional IOPs. The four distinct groups obtained in this way best reflect the existing differential patterns of multidimensional opportunity in Europe. Finally, the Shapley decomposition analysis reveals the primary importance of parental background characteristics in shaping multidimensional inequality of opportunity in Europe.

The paper extends the broad literature on equality of opportunity by providing tools that have been missing and called for (Bourguignon et al. 2007b). On a formal level, the model presented in the paper is related to the framework proposed by Peragine (2004), who characterizes classes of equal opportunity welfare functions and obtains á la Lorenz dominance conditions for the ranking of income distributions. We generalise Peragine's results to a multidimensional setting and obtain appropriate multidimensional conditions, but we do much more as we develop inequality measures. This step is crucial because dominance criteria are often inconclusive. Our axiomatic approach is also related to the work of Maasoumi (1986) and Tsui (1995, 1999), who address the problem of measuring (outcome) inequality in the case of multidimensional distributions. In particular, Tsui (1999) proposes multidimensional generalisations of the Pigou-Dalton transfer principles, such as Uniform Pigou-Dalton Transfer or Uniform Majorization, and also considers transfers that change the dependence structure of the distribution, the so-called correlation-increasing switches. Following Tsui (1999), we adapt these properties to our context of inequality of opportunity. Formally, we obtain an extension of his class of measures to the case of several distinct subgroups. A related body of empirical work consists of several papers that examine the existence of unequal opportunities for multiple outcomes. Bourguignon et al. (2007a) analyze income and schooling outcomes in Brazil. Ferreira and Gignoux (2011) focus on different income measures. Peragine and Serlenga (2008) analyze university graduation outcomes and later life earnings. Ferreira and Gignoux (2010) study Pisa scores in reading, mathematics and science in Turkey and Paes de Barros et al. (2009) education and access to basic services such as sanitation in Latin American and Asia. These papers have in common that they treat the issue of multidimensionality in the same oversimplified way: outcomes are analysed separately, neglecting any interdependencies between them. The methods developed in this paper address this shortcoming.

The paper is structured as follows. It begins (Section 2) with the introduction of the analytical framework and basic definitions and properties. Section 3 contains the characterisation theorems for classes of social welfare functions. and links them to dominance conditions. Based on these results, in Section 4, inequality measures are derived. Section 5 applies the developed methods to the EU SILC data. In Section 6, which concludes the paper, we mention several possible extensions to the current framework. Appendix A contains proofs and Appendix B robustness checks for the empirical analysis.

## 2 Definitions and axioms

The underlying conceptual model of the theory of equality of opportunity is the following. We have a society consisting of $N$ individuals. Each individual is completely described by an outcome $x$, which is a multidimensional variable consisting of $k$ attributes, and by a set of traits. Traits can be divided into traits beyond the responsibility of individuals, called circumstances, and traits for which the individual is fully responsible, called effort. Formally, circumstances are represented by a set variable $O \in \mathcal{O}$;[4] and effort is represented by $w \in \Theta \subseteq \mathbb{R}_+$. The elements of $\mathcal{O}$ are sorted according to ordering $\prec$, that is $O_i \prec O_{i+1}$ for $i \in \{1, 2, \ldots, n-1\}$. Outcome is generated by a function $g : \mathcal{O} \times \Theta \to \mathbb{R}^k_+$ that assigns individual out-

[4] $\mathcal{O}$ is a general space, not necessarily numbers, but also descriptions e.g., as in the case of self-reported ordered variables.

comes to combinations of circumstances and effort: $x = g(O, w)$.[5] We assume that $g$ is monotone with respect to the ordering on circumstances and with respect to effort, where monotonicity means that $g$ increases according to some ordering on the space of $k$ outcomes $\mathbb{R}^k_+$. For example, depending on additional properties, this ordering could be that $g$ increases coordinate by coordinate, or less restrictively, that $g$ increases as a simple sum of $k$ attributes.[6]

In the ex-ante approach, the focus is on the prospects measured by the set of opportunities that individuals face *before* they exert particular effort. The opportunity set is assumed to be the set of all outcomes in a given circumstance group.[7] Therefore, effort is not treated explicitly; it is only the source of underlying heterogeneity within the circumstance group. The outcome distribution is the primitive of the model that is used for the purposes of measurement. Formally speaking, it is the outcome distribution that arises from $g$ for a given $\theta$ and all possible values of $\omega$, assuming that every individual has access to the same degrees of effort. We will now focus on this distribution.

We partition the society into $n$ ordered types $1 < \ldots < n$, a type $h$ being a set of individuals with the same circumstances. Let $N_h$ be the number of people in type $h$. We have $\Sigma_{h=1}^n N_h = N$. Let $X^h \in M_{N_h \times k}$ denote outcome distribution within type $h$. As mentioned, it also defines the opportunity set available to individuals in type $h$. $M_{N_h \times k}$ is the set of such matrices with values in $\mathbb{R}_+$. $X^h_{ij}$ is the value of $j$-th attribute for an $i$-th individual who belongs to type $h$ and $X^h_{i.}$ is his or her vector of attributes. Let $X \in M_{N \times k}$ represent outcome profile of the whole society and $M_{N \times k}$ be the set of all possible outcome profiles.

Figure 1 shows how type matrices $X^h$ form a full outcome matrix $X$. Matrices $X^h$ are arranged into matrix $X$ in an increasing order, that is rows $1, ..., N_1$ of $X$ are rows of $X^1$, rows $N_1+1, ..., N_1+N_2$ are rows of $X^2$ and so on. Finally, let $X_\mu$ be a matrix of type-means. It represents the case of perfect within-type equality, but *not* between-type equality. The latter is denoted $X^\mu$; it is a matrix of population

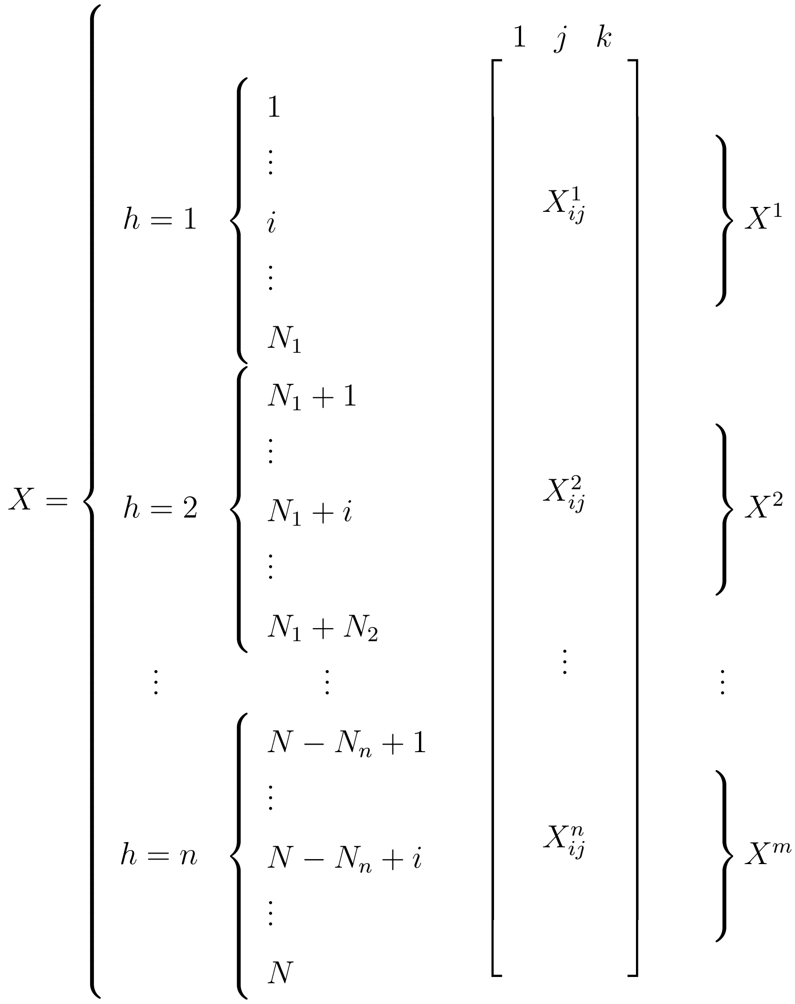
[5]In other words, given a combination of effort and circumstances, function $g$ completely determines the outcomes i.e. all individuals with a given effort-circumstance bundle share the same outcomes. As already mentioned, richer models of EOp also include "luck component" (Lefranc et al. 2009).

[6]We use the words "outcomes, attributes, dimensions" interchangeably.

[7]A particular position in this distribution depends on the exerted effort and is a building block of the ex-post approach (Roemer 1996).

means. They have the same dimension as matrix $X$, that is, $X_\mu, X^\mu \in M_{N\times k}$. We write $(X_\mu)^h_{ij}$ (resp. $(X^\mu)^h_{ij}$) when we mean a value for individual $i$ in outcome $j$ in type $h$ of matrix $X_\mu$ (resp. $X^\mu$) remembering that these values are the same for all individuals in a type ($X_\mu$) and in case of $X^\mu$ also across types.[8] If we mean the whole vector of outcomes, we put $(X_\mu)^h_{i\cdot}$ (resp. $(X^\mu)^h_{i\cdot}$).

Figure 1: Structure of matrix $X$




We are interested in the ranking of outcome matrices and to this end we examine a binary preference relation of an opportunity egalitarian social planner. We assume that this relation is a continuous ordering, hence it is represented via a continuous monotone social evaluation function $W : M_{N\times k} \to \mathbb{R}$. $I : M_{N\times k} \to [0, 1]$ is a corresponding inequality index. Formally, the relationship between $W$ and $I$ is defined in the following way.

[8]For this reason, sometimes we set $i = 1$.

**Definition 1.** *We say that an inequality measure $I$ is induced by a welfare function $W$ if $I(X) = 1-\delta(X)$ where $\delta(X) \in [0,1]$ satisfies equation $W(X) = W(\delta(X)X^{\mu})$.*

This is the so called Atkinson-Kolm-Sen approach (Atkinson, 1970; Kolm 1969; Sen, 1973) to constructing inequality measures which ensures that they are normatively significant, meaning that under certain restrictions (e.g. equality of means) $W(X) \leq W(Y) \iff I(X) \geq I(Y)$. We will now delineate the axioms of EOP that characterize $W$. We start by defining multidimensional inequality-reducing transfers. These definitions will then be used in subsequent properties imposed on $W$.

**Definition 2.** ***Pigou-Dalton Transfer between Types (PDTT)*** *Let $\theta_s \prec \theta_r$. For $X, Y \in D$, we say that $X$ is obtained from $Y$ by Pigou-Dalton Transfer between Types (PDTT) if $X^h = Y^h$ for $h \notin \{r, s\}$, $X^r_{i.} = Y^r_{i.}, X^s_{i.} = Y^s_{i.}$ for $i \notin \{p, q\}$ and for some individuals $p, q$ belonging to, respectively, types $r, s$, we have $X^r_{pj} = Y^r_{pj}(1-\varepsilon) + Y^s_{qj}\varepsilon$ and $X^s_{qj} = Y^r_{pj}\varepsilon + Y^s_{qj}(1-\varepsilon)$, where $\varepsilon \geq 0$ and after the transfer still $\theta_s \prec \theta_r$.*

PDTT is a transfer between individuals $p, q$ from, respectively, types $r, s$. Other elements of matrices $X, Y$ are fixed. It is a uniform PD majorization (Tsui 1995, Marshall and Olkin 1979) which ensures that two individuals are brought closer together via a transfer of the same proportionate amount on each dimension. Furthermore, it preserves the ordering of types. $\theta_s \prec \theta_r$ means that type $r$ is better than type $s$ according to some ordering $\prec$ on the space of circumstances. What is this ordering in practice? The standard approach in the literature is to order types based on outcomes e.g., better type is the one that has a higher mean income. We follow this approach and consider two such orderings: (i) a better type is the one that has higher mean value of each outcome, and (ii) a better type is the one that has higher average utility of outcomes, as measured by a class of utility functions specified in our results. Therefore, PDTT is such that it reduces distance between two persons and thus two types on each outcome, but it preserves the ordering of mean outcomes or type utilities. When $r = s$ the transfer takes place within type.

**Definition 3.** ***Pigou-Dalton Transfer (PDT)*** *For $X, Y \in D$, we say that $X$ is obtained from $Y$ by Pigou-Dalton Transfer (PDT) if $X$ is obtained from $Y$ by PDTT for $r = s$.*

In a multidimensional setting welfare and inequality are not only about the spread, but also about dimensions' dependence. This is reflected in the definition of a correlation-increasing transfer (Boland and Proschan 1988), where between two individuals one is assigned maximum values of each outcome and the other is assigned minimum values. Such an operation clearly increases correlation between outcomes. Although marginal outcome distributions do not change, there is now "more joint risk", i.e., there is higher likelihood that a given individual occupies higher or lower positions in several dimensions. This is formally defined below as Correlation Increasing Transfer (CIT) which acts within type.

**Definition 4.** ***Correlation-Increasing Transfer (CIT)*** *For all $X, Y \in D$, we say that $Y$ is obtained from $X$ by Correlation-Increasing Transfer (CIT), if $X^h = Y^h$ for $h \neq l$, $Y^l_{ij} = X^l_{ij}$ for $i \neq \{p, q\}$ and for some individuals $p, q$ belonging to type $l$, we have $Y^l_{pj} = \max(X^l_{pj}, X^l_{qj})$, $Y^l_{qj} = \min(X^l_{pj}, X^l_{qj})$ for all $j$.*

Definitions 3 and 4 are the basis of the axioms imposed on function $W$.

**MONOTONICITY (MON).** For all $X, Y \in D$, if $X^h = Y^h$ for all $h \neq l$, $X^l_{pj'} = Y^l_{pj'} + \varepsilon$ with $\varepsilon \geq 0$ and $X^l_{ij} = Y^l_{ij}$ for $(i, j) \neq (p, j')$, then $W(Y) \leq W(X)$.

**ADDITIVITY (ADD).** There exist functions $U^h : \mathbb{R}^k \to \mathbb{R}$, for all $h \in \{1, 2, \ldots, n\}$, assumed to be twice differentiable in the variable $X^h_{i.}$, such that $W(X) = \Sigma^n_{h=1} \Sigma^{N_h}_{i=1} U^h(X^h_{i.})$ for all $X \in D$.

**INEQUALITY NEUTRALITY WITHIN TYPES (INWT).** For all $X, Y \in D$, if $X$ is obtained from $Y$ by PDT, then $W(X) = W(Y)$.

For all $X, Y \in D$, if $X$ is obtained from $Y$ by CIT, then $W(X) = W(Y)$.

**INEQUALITY AVERSION WITHIN TYPES (IAWT).** For all $X, Y \in D$, if $X$ is obtained from $Y$ by PDT, then $W(Y) \leq W(X)$.

For all $X, Y \in D$, if $Y$ is obtained from $X$ by CIT, then $W(Y) \leq W(X)$.

**INEQUALITY AVERSION BETWEEN TYPES (IABT)** For all $X, Y \in D$, if $X$ is obtained from $Y$ by PDTT, then $W(Y) \leq W(X)$.

MON and ADD are widely assumed in the literature. MON states that welfare function does not decrease following an increase in the value of a single attribute for a single individual. ADD means that $W$ standard utilitarian welfare functions i.e. a sum of individual utilities. Please note that an individual utility function is a function of many variables.[9] INWT, IAWT and IABT are conceptual and specific to the EOP framework. The first two reflect different versions of the reward principle: INWT is an expression of utilitarian reward and states that $W$ is invariant to inequality-reducing transformations within type, i.e., PDT and CIT. On the contrary, IAWT is an expression of inequality-averse reward and thus of aversion to inequality within types. IABT expresses the ex-ante principle of compensation (see e.g. Peragine 2004; Fleurbaey and Peragine 2013), which states that inequality-reducing transfer between types increases welfare.

# 3 Welfare functions and dominance conditions

In this section we study two classes of social welfare functions and for each of them we give a dominance condition which implements the ordering prescribed by a given class. The usefulness of such characterization results is that they allow to replace the unobservable class of welfare functions with conditions that can be implemented using data on individual outcomes. We start with the class of social welfare functions which combines **ex ante compensation and utilitarian reward**

$$\mathcal{W}^{AOEN} = \{W|\text{MON}, \text{ADD}, \text{INWT}, \text{IABT}\}$$

and the assciated class of utility functions

$$\mathcal{U}^{AOEN} = \{U|\text{Increasing}, \text{Type-Concave}, \text{Linear}\}.$$

AOEN means "additive opportunity egalitarian neutral", where the last term comes from neutrality embedded in INWT. By Increasingness we understand that $U^h$ is increasing with respect to attributes. Type-Concavity means that worse type is more important in welfare evaluation; the better the type the lower its first derivative.

[9] $U^h$ is the same for individuals within type, which means that symmetry within types is assumed: $W$ is invariant with respect to permutations of individual bundles *within* type. It is not, however, invariant with respect to permutation of individual bundles between types as in standard symmetry, because type membership is important in EOP theory.

**Definition 5.** ***Type-Concavity*** *Function* $U^h : \mathbb{R}^k \to \mathbb{R}$ *is type-concave if for* $\theta_s \prec \theta_r$*, we have* $dU^s/dX \geq dU^r/dX$.

INWT brings us asymptotically to matrix $X_\mu$ characterized by perfect within-type equality. Indeed, in case of INWT a social decision maker does not care about the distribution of outcomes within a type. This is expressed formally in the following lemma.

**Lemma 1.** *Let* $X \in D$ *and* $W$ *satisfies INWT. Then* $W(X) = W(X_\mu)$*, where* $(X_\mu)^h_{ij} = \frac{\Sigma_{i=1}^{N_h} X^h_{ij}}{N_h}$.

We now define Component-wise Generalized Lorenz Dominance. It performs standard Generalized Lorenz Dominance for each attribute separately i.e. it works on vector columns of matrix $X$. Please note that total sums in columns in $X$ and $Y$ may differ.

**Definition 6.** ***Component-wise Generalized Lorenz Dominance (CGLD)*** *For all increasingly ordered (within columns) matrices* $X, Y \in M_{N\times k}$*, we say that* $X$ *Component-wise Generalized Lorenz dominates* $Y$ *if and only if*

$$X \succeq_{CGLD} Y \iff \Sigma_{i=1}^{l} X_{ij} \geq \Sigma_{i=1}^{l} Y_{ij} \quad \forall_{1\leq l\leq N} \forall_{1\leq j\leq k}$$

In Theorem 1 we relate CGLD to a class of functions $\mathcal{W}^{AOEN}$. This theorem is a multidimensional analog of Theorem 1 in Peragine (2004).

**Theorem 1.** *For* $X, Y \in D$ *we have*

$$X_\mu \succeq_{CGLD} Y_\mu \iff W(Y) \leq W(X) \quad \textit{for all} \quad W \in \mathcal{W}^{AOEN},$$

The precise form of $W$ follows from Theorem 1 and Lemma 1. Functions in the class $\mathcal{W}^{AOEN}$ are weighted sums of type means with higher weights assigned to worse types to preserve inequality aversion between types.

**Corollary 1.** $W \in \mathcal{W}^{AOEN}$ *are of the following form*

$$W(X) = Nb + \sum_{h=1}^{n} \sum_{i=1}^{N_h} \sum_{j=1}^{k} a^h_j X^h_{ij} = Nb + \sum_{h=1}^{n} N_h \sum_{j=1}^{k} a^h_j (X_\mu)^h_{1j}, \tag{1}$$

*where* $a^h_j \geq a^{h+1}_j \geq 0$ *for all* $j$*. In what follows we omit constants* $Nb$*.*

Theorem 1 states that if for each attribute the type-means distribution of $X$ Generalized Lorenz dominates the type-means distribution of $Y$, then for all welfare functions in the class $\mathcal{W}^{AOEN}$ $X$ is more opportunity egalitarian than $Y$. The converse is also true. Theorem 1 gives a simple prescription of how to implement unanimous welfare rankings for the case of neutrality. First partition the society into types, then for each attribute compute the distribution of type means. Finally, increasingly order the columns of this distribution and apply Generalized Lorenz dominance to each column. The result is restrictive – each dimension is treated separately and therefore the obtained ranking has a meaningful interpretation only when a better type has a higher mean than a worse type on each dimension. This restrictiveness is driven by the powerful axiom INWT.

We will now study welfare functions that combine **ex ante compensation and inequality averse reward**. The Component-Wise Generalized Lorenz Dominance characterized in the previous section is non-welfarist in the sense that the evaluation of distributions depends directly on the values of attributes. Different authors (e.g. Maasoumi 1986, Dardanoni 1992) propose a welfarist approach to multidimensional aggregation. Namely, an individual well-being derived from attributes is first evaluated through a utility function. This leads to the utility-based majorization such as the one below.

**Definition 7.** ***Welfarist Generalized Lorenz Dominance (WGLD)*** *Let $X, Y \in D$ and $u_h^X$ be defined as $u_h^X = \sum_{i=1}^{N_h} U^h(X_{i.}^h)$. For all increasingly ordered vectors $u_h^X, u_h^Y$, we say that $X$ Welfarist Generalized Lorenz dominates $Y$ in a class $\mathcal{F}$ of functions $U_h$ if and only if*

$$X \succeq_{LD(\mathcal{F})} Y \iff \sum_{h=1}^{l} u_h^X \geq \sum_{h=1}^{l} u_h^Y \quad \forall_{l=1,\ldots,n} \forall_{U^h \in \mathcal{F}}.$$

WGLD first aggregates individual utilities within type, then compares partial sums of such within-type aggregate utility vectors. Here we make the dependence on $\mathcal{F}$ explicit to underlie that WGLD can be applied to different classes of utility functions. The welfarist GLD criterion will enable us to compare distributions by welfare orderings in which a policymaker cares about both inequality between types (IABT) and inequality within type (IAWT).

We define the following class of welfare functions, where AOEA means "additive opportunity egalitarian averse"

$$\mathcal{W}^{AOEA} = \{W|\text{MON}, \text{ADD}, \text{IAWT}, \text{IABT}\}$$

and the associated class of utility functions

$$\mathcal{U}^{AOEA} = \{U|\text{Increasing}, \text{Type-Concave}, \text{Concave}, \text{Submodular}\}.$$

**Definition 8.** ***Submodularity*** *Function $U^h$ is submodular if $U^h(X^h_{p.})+U^h(X^h_{q.}) \geq U^h(X^h_{p.} \wedge X^h_{q.})+U^h(X^h_{p.} \vee X^h_{q.})$ where $X^h_{p.} \wedge X^h_{q.}$ is a vector of elements $\max\{X^h_{pj}, X^h_{qj}\}$ and $X^h_{p.} \wedge X^h_{q.}$ is a vector of $\min\{X^h_{pj}, X^h_{qj}\}$.*

Note that $X^h_{p.} \vee X^h_{q.}$ denotes element-wise maximum and $X^h_{p.} \wedge X^h_{q.}$ denotes element-wise minimum. The function is submodular if it attains lower value for the distribution such that between two individuals, one has lower (or higher) value than the other for *each* attribute. Submodularity reflects that association between dimensions matters, and if there is more of it the utility is lower.

Theorem 2 combines welfare functions and dominance criterion. WGLD applied to the class $\mathcal{U}^{AOEA}$ is an implementable criterion for "additive opportunity egalitarian averse" welfare functions.

**Theorem 2.**

$$X \succeq_{WGLD(\mathcal{U}^{AOEA})} Y \iff W(Y) \leq W(X) \quad \textit{for all} \quad W \in \mathcal{W}^{AEOA}$$

Theorem 2 states that in order to compare distributions of outcomes in terms of welfare functions that are monotone, additive and averse with respect to both inequality within and between types, one can apply WGLD to type-aggregate utilities, where utility functions are increasing, type-concave, concave and submodular. The reverse is true as well, namely, WGLD for such utility functions is the largest (in the sense of inclusion) ordering on $D$ consistent with the class $\mathcal{W}^{AEOA}$.

The following corollary states what is clear from the proof of Theorem 2, namely, that if goods are complements not substitutes, the theorem still holds but for supermodular utility functions.

**Corollary 2.** *In Definition of IAWT we change the sign of function $W$ i.e. for all $X, Y \in D$ if $X$ is obtained via CIT from $Y$, then $W(Y) \leq W(X)$ and in Definition 8 we change the sign of inequality i.e. $U^h(X^h_{p.}) + U^h(X^h_{q.}) \leq U^h(X^h_{p.} \wedge$*

$X^h_{q.}) + U^h(X^h_{p.} \vee X^h_{q.})$ *(i.e. function $U^h$ is supermodular). Then Theorem 2 holds for*

$$\mathcal{U}^{AOEA'} = \{U|\text{Increasing, Type-Concave, Concave, Supermodular}\}.$$

# 4 Inequality of opportunity measures derived from welfare functions

In this section we first define the inequality of opportunity measures and then characterize the corresponding measures for each class of welfare functions considered in Section 3.

**Definition 9.** *$I$ is an inequality of opportunity (IOP) measure if it satisfies the following properties:*

1. *$I$ is continuous.*
2. *$I(X^\mu) = 0$ i.e. $I$ is zero for perfect equality.*
3. *$I(Y) \leq I(X)$ if $X$ is PDTT of $Y$.*

*Measure $I$ is relative if additionally $I(XC) = I(X)$ for a diagonal matrix $C$. If $C$ has all elements on the diagonal equal, then then we say that $I$ is weakly relative.*

In what follows we derive IOP measures for both $\mathcal{W}^{AOEN}$ and $\mathcal{W}^{AOEA}$ starting with $\mathcal{W}^{AOEN}$ i.e. **ex ante compensation and utilitarian reward**. Before we characterize the IOP measures for the class $\mathcal{W}^{AOEN}$, please note that the second point of Definition 9 does not state that $I$ is zero only for $X_\mu$. In fact, for welfare functions satisfying INWT, the corresponding inequality measure $I$ is zero for any $X$ which has total sums equal everywhere to that of $X^\mu$. That is, if the index is indifferent to inequality within type, the total sum of each attribute can be distributed in any way within type and the index should not change. It is not straightforward that inequality measures induced from welfare functions $W \in \mathcal{W}^{AOEN}$ according to Definition 1 satisfy Definition 9, but with slight restriction, namely $W(X) \neq 0$, this is indeed the case.

**Theorem 3.** *Let $W \in \mathcal{W}^{AOEN}$ with $W(X) \neq 0$ for any $X \in D$. Then we have the following set of results:*

1. *$I$ is given by* $1-\frac{W(X)}{W(X^{\mu})}=1-\frac{\sum_h\sum_i\sum_j a_j^h X_{ij}^h}{\sum_{h=1}^n N_h\sum_{j=1}^k a_j^h (X^{\mu})_{1j}^h}$.

2. *$I$ is given by* $1-\frac{W(X_{\mu})}{W(X^{\mu})}=1-\frac{\sum_{h=1}^n N_h\sum_{j=1}^k a_j^h (X_{\mu})_{1j}^h}{\sum_{h=1}^n N_h\sum_{j=1}^k a_j^h (X^{\mu})_{1j}^h}$.

3. *$I$ is a weakly relative inequality measure, but not a relative measure.*

Theorem 3 states that the index related to the class of welfare functions $\mathcal{W}^{AOEN}$ is one minus the weighted sum of type-means for each dimension normalized by the highest amount of welfare achievable (point 2 in the Theorem). Types' weights $a_j^h$ are as in Corollary 1, that is, on each dimension $j$, we have $a_j^h \geq a_j^{h+1} \geq 0$. Higher weights are assigned to types that have lower mean on each dimension. This measure is a weakly relative measure, that is, it does not change when all attributes are scaled by the same factor, but it is not invariant when each attribute is scaled by its mean. Linearity does not allow for a stronger form of scale invariance.

For $\mathcal{W}^{AOEA}$, i.e. **ex ante compensation and inequality averse reward**, it is not possible to derive a specific formula of IOP measure, because this class is too general. However, we can find a subclass for which this can be done. This subclass has an additional property.

**RATIO SCALE INVARIANCE (RSI)** For all $X,Y \in D$ and diagonal matrix $C$ we have $W(X)=W(Y) \iff W(XC)=W(YC)$.

Recalling Definition 9, we can see that the considered subclass of welfare functions consists of exactly those $W \in \mathcal{W}^{AOEA}$ for which $I$ is relative. The following theorem is our main result.

**Theorem 4.** *Let $W \in \mathcal{W}^{AOEA}$ and $W$ satisfies RSI. Then we have the following set of results:*

1. *$I$ is a relative inequality measure.*

2. $U^h(X)=a_h\prod_{j=1}^k\left(X_{ij}^h\right)^{r_j}$, *where* $a_h<0, r_j<0$.

3. *$I(X)$ is given by*

$$I(X)=1-\left(\sum_{h=1}^n w_h\frac{U^h\big((X_{\mu})_{1\cdot}^h\big)}{U^h\big((X^{\mu})_{1\cdot}^h\big)}\right)^{\frac{1}{\sum_{j=1}^k r_j}} \tag{2}$$

*where* $w_h=\frac{\delta_h(X)N_h a_h}{\sum_{h=1}^n N_h a_h}$ *and* $\delta_h(X)$ *is*

$$\delta_h(X)=\left[\frac{1}{N_h}\sum_{i=1}^{N_h}\frac{U^h\big(X_{i\cdot}^h\big)}{U^h\big((X_{\mu})_{1\cdot}^h\big)}\right].$$

4. *Given point 2, we simplify*

$$I(X) = 1 - \left( \sum_{h=1}^{n} \frac{a_h \sum_{i=1}^{N_h}}{\sum_{h=1}^{n} N_h a_h} \prod_{j=1}^{k} \left( \frac{X_{ij}^h}{(X^\mu)_{1j}^h} \right)^{r_j} \right)^{\frac{1}{\sum_{j=1}^{k} r_j}} \tag{3}$$

*and*

$$\delta_h(X) = \left[ \frac{1}{N_h} \sum_{i=1}^{N_h} \prod_{j=1}^{k} \left( \frac{X_{ij}^h}{(X_\mu)_{1j}^h} \right)^{r_j} \right].$$

5. *When* $\delta_h(X_\mu) = 1$ *for all* $h$*,* $I$ *takes into account only between type inequality and is given by*

$$I(X) = 1 - \left( \sum_{h=1}^{n} \frac{N_h a_h}{\sum_{h=1}^{n} N_h a_h} \frac{U^h \left( (X_\mu)_{1\cdot}^h \right)}{U^h \left( (X^\mu)_{1\cdot}^h \right)} \right)^{\frac{1}{\sum_{j=1}^{k} r_j}}, \tag{4}$$

*or equivalently,*

$$I(X) = 1 - \left( \sum_{h=1}^{n} \frac{N_h a_h}{\sum_{h=1}^{n} N_h a_h} \prod_{j=1}^{k} \left( \frac{(X_\mu)_{1j}^h}{(X^\mu)_{1j}^h} \right)^{r_j} \right)^{\frac{1}{\sum_{j=1}^{k} r_j}}. \tag{5}$$

Let us now explain the statements contained in Theorem 4 one by one. Point 1 of the theorem states that the inequality of opportunity index is relative; this is a direct consequence of RSI imposed on welfare functions in $\mathcal{W}^{AOEA}$. For this subclass of welfare functions, thanks to the results of Aczél (1988) and Tsui (1995) the precise functional form of individual utility functions (Cobb-Douglas) can be derived (point 2). Then (point 3), the obtained (via Definition 1) inequality of opportunity measure is a weighted sum of normalized types' utilities i.e. $\frac{U^h\left((X_\mu)_{1\cdot}^h\right)}{U^h\left((X^\mu)_{1\cdot}^h\right)}$. The normalization factor is the value of highest welfare which is obtained for perfectly equal distribution $X^\mu$.[10] The weights in this weighted sum, $\omega_h$, are types' population shares $\frac{N_h a_h}{\sum_{h=1}^{n} N_h a_h}$ times the value of *equality* within type as measured by Tsui (1995) equality indices ($\delta_h$). Thus both within and between type inequality can be easily distinguished. The functional form of utility function (point 2) can be utilized to obtain a more precise form of $I$ (point 4). When there is no variation within type ($\delta_h = 1$), $I$ focuses solely on inequality between types and as such takes into account only how distribution $X_\mu$ compares to distribution $X^\mu$

[10] As noted by Bourguignon (1999, p. 478) "maximizing social welfare under the constraint of fixed total resources of attributes. . . requires to give each individual the average available quantity of attributes". An alternative normalization, by mean well-being, is used in the so-called two-stage approach to measuring multivariate inequality. For more, pleasee see Aaberge and Brandolini (2015, p. 197). In the univariate case the two normalizations coincide.

(point 5). In the empirical application we call this measure "pure between type" to distinguish from "full measure" (2).

Let us now explain the parameters used in $I$: type weights $a_h$ and dimension weights $r_j$ which are both chosen by the researcher. There are both negative which makes the measure convex and supermodular. Convexity ensures that the measure decreases after transfers that reduce dimensional spreads (PDT) and supermodularity ensures that it increases after transfers that increase dimensions' correlation (CIT). Thus, these properties ensure that we indeed have a measure of multidimensional inequality. Going into more detail, the more negative the dimension weight $r_j$, the more convex the measure along outcome $j$ and the more important the outcome $j$ is in the inequality assessment. Viewed from a different angle, $r_j = 1-\epsilon_j$, where $\epsilon_j$ is an inequality aversion parameter for dimension $j$ (see e.g., Atkinson, 1970).[11] In his seminal work, Atkinson (1970) arbitrarily set the income inequality aversion parameter equal to 1, 1.5 and 2. Since then, empirical literature tried to estimate it from tax schedules which are assumed to reflect societal preferences towards redistribution in democratic societies (see e.g. Young 1990, Aristei and Perugini 2016). The obtained estimates range between 1 and 2 depending on the country and time period of interest. In the empirical section, we choose parameters from this range in the baseline scenario and sensitivity analysis. While $r_j$s reflect how much the social planner values respective outcomes, type weights $a_h$ reflect how much the social planner values respective types. To ensure that inequality aversion between types is preserved (IABT), higher type weights $a_h$ is assigned to types that are "worse". Here, consistent with Lorenz dominance criterion (Definition 7) a worse type is the one that has lower type utility. In particular, the assignment of weights is the following: $a_h \geq a_{h+1}$ for $-\frac{1}{N_h}\sum_{i=1}^{N_h}\prod_{j=1}^{k}\left(X_{ij}^{h}\right)^{r_j} \leq -\frac{1}{N_{h+1}}\sum_{i=1}^{N_{h+1}}\prod_{j=1}^{k}\left(X_{ij}^{h+1}\right)^{r_j}$. Type weights decrease with type rank in terms of its average utility. In the empirical section, as a baseline scenario we choose type weights that decrease linearly and for a sensitivity analysis we consider weights that are concave or convex in type ranks.

[11] This comes from the fact that Tsui (1995) is a multidimensional extension of Atkinson's (1970) inequality index, so dimension weights have this interpretation. As $\epsilon_j$ rises, from a welfare point of view it becomes more efficient to reduce inequality at the bottom of the distribution than at the top, because for with higher degree of inequality aversion a transfer at the bottom reduces inequality by more than the same transfer at the top.

To sum up, the procedure for computing IOP is the following: firstly, order types based on average utility as measured by utility function described in point 2 of Theorem 2 and assign the chosen type weights consistently with this ranking; secondly, compute IOP according to points 3 or 4. One important thing to note is that unidimensional inequality indices (which are then Atkinson's inequality of opportunity indices) follow easily from the measure by putting the respective dimensional weight $r_j$ equal to 0. Unidimensional IOP measures are thus naturally embedded in the multidimensional measure. Finally, given Atkinson-Kolm-Sen construction which we use (Definition 1), the measure has a natural interpretation. For example, a value of 0.25 means that the current inequality of opportunity imposes a welfare cost of 25% mean value of each outcome. In other words, if there was perfect equality of opportunity, society would achieve the same level of welfare with only 75% of currently available resources of each outcome.

# 5 Empirical application

We will now illustrate the methods developed in the paper with an empirical application. To this end we use the latest wave (year 2019) of EU-SILC (European Union Statistics on Income and Living Conditions). EU-SILC contains information on individual life outcomes such as income, education and health as well as family background for a wide range of European countries. It is the official reference source for comparative statistics on income distribution and social inclusion in Europe. Variable definitions and collection procedures are the same in each country. Our analysis is based on the 2019 ad-hoc module "Intergenerational transmission of disadvantages, household composition, and the evolution of income". This module contains detailed information on family background characteristics thus providing us with harmonized individual-level data on outcomes and circumstances for 29 European countries.

As outcomes we take three variables, namely, income, health, and education. Summary statistics are provided in Table 1. Income is annual disposable household income scaled using the modified OECD equivalence scale and expressed in 2019 Euro. Education is the highest level of educational attainment according to the six levels of the International Standard Classification of Education (ISCED). Health variable is self-reported health status measured on a 5-point Likert-scale ranging from very bad to very good.

Table 1: Summary statistics for outcome variables

| | Mean | SD |
|---|---|---|
| Income (2019 EUR) | 15480.08 | 8375.65 |
| Self-reported health (1-very bad; 5-very good) | 3.98 | 0.79 |
| Education degree (6 levels ISCED) | 4.98 | 1.33 |

Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Circumstance variables were derived from an ad-hoc module. Most of these variables refer to the situation in the household when respondent was around 14 years old. We consider six circumstance variables: biological sex (1 for male), parental education (1 - at least one parent has higher education), family composition (1 - both parents are present in the household), degree of urbanisation (1 - City (more than 100 000 inhabitants), 2 - Town or suburb (10 000 to 100 000 inhabitants) - 3 Rural area, small town or village (less than 10 000 inhabitants)), financial situation of the household when respondent was 14 years old (1 - very bad, bad or moderately bad, 2 - moderately good, 3 - good or very good) and parental occupation (1 - low: craft and related trades workers, plant and machine operators and assemblers, elementary occupations; 2 - medium: armed forces occupations, technicians and associate professionals, clerical support workers, service and sales workers, skilled agricultural, forestry and fishery workers; 3 - high: managers, professionals). Summary statistics are provided in Table 2. Taking into account missing data on individual income and parental education gives 150226 observations.

Figures 2 - 5 contain base results: IOP estimates for joint and unidimensional outcomes. Type weights are negative and linear (-maximum number of types $+h$) and dimension weights are $r_j = -0.5$ for all $j$.[12] Countries with the highest income inequality of opportunity are Southeastern countries such as Romania (0.279), Bulgaria (0.277), Serbia (0.220) and Baltics: Lithuania (0.236) and Latvia (0.228). Here income is the most influenced by past conditions. On the other end of the spectrum are rich European countries such as Switzerland (0.069),

[12]In Appendix B we run several robustness checks varying both sets of weights which does not change our main conclusions.

Table 2: Summary statistics for circumstance variables

| | Mean | SD |
|---|---|---|
| Sex (1-male) | 0.490 | 0.499 |
| Parental education (1-at least one parent with higher degree) | 0.557 | 0.496 |
| Parental occupation (1-low; 2-medium; 3-high) | 1.861 | 0.722 |
| Family composition (1-both parents present in household) | 0.899 | 0.301 |
| Degree of urbanisation (1-city; 2-town; 3-rural) | 1.810 | 0.798 |
| Family financial situation when 14 years old (1-bad; 3-good) | 2.172 | 0.757 |

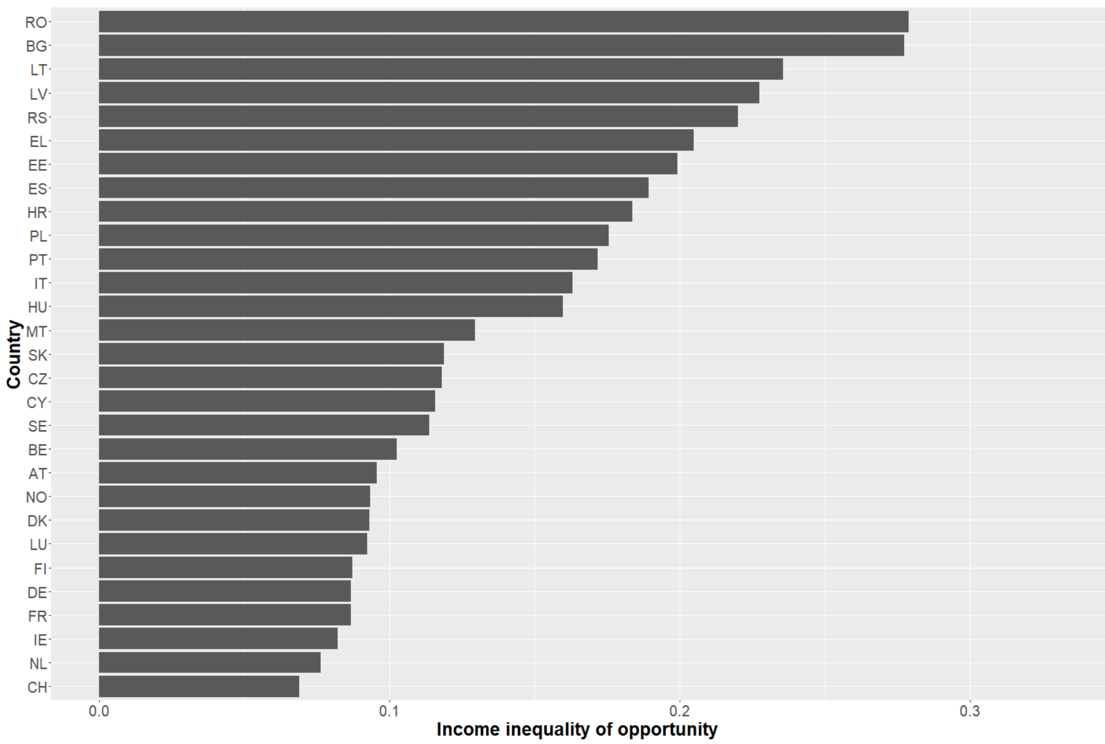
Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Netherlands (0.076), Germany (0.087) and France (0.087), Luxembourg (0.093) and Scandinavian countries (Finland - 0.087, Norway - 0.061, Denmark - 0.059). Interestingly, this ranking reverses for health opportunity. Here, Denmark (0.123), Norway (0.98), Luxembourg (0.91) and Switzerland (0.075) emerge as the most unequal countries, whereas Romania (0.038) and Bulgaria (0.046) are among the least unequal together with Italy (0.034) and Malta (0.037). Furthermore, Luxembourg (0.092) and Norway (0.79) are amongst the most unequal countries in terms of educational opportunity, with Portugal being the most unequal (0.103) and other southern countries such as Greece (0.083) and Spain (0.078). Poland (0.072) stands out from Eastern countries which are the most egalitarian for educational attainment: Czech Republic (0.019), Slovakia (0.022), Lithuania (0.029) and Latvia (0.030).

From these comparisons, we can see that unidimensional rankings diverge and the impact of circumstances is different depending on the outcome. Joint treatment of the outcomes resolves these ambiguities. According to our joint measure, the most jointly unequal countries are Bulgaria (0.143), Romania (0.133) and Portugal (0.125). Luxembourg, Norway and Denmark occupy positions in the middle. The most equal countries in terms of joint opportunities are Switzerland (0.052), Finland (0.054), Netherlands (0.061) and Germany (0.063). Joint inequality of opportunity is highly correlated with income IOPP (0.84) although not perfectly, much less with educational IOPP (0.45) and negatively correlated with health IOPP (-0.19).

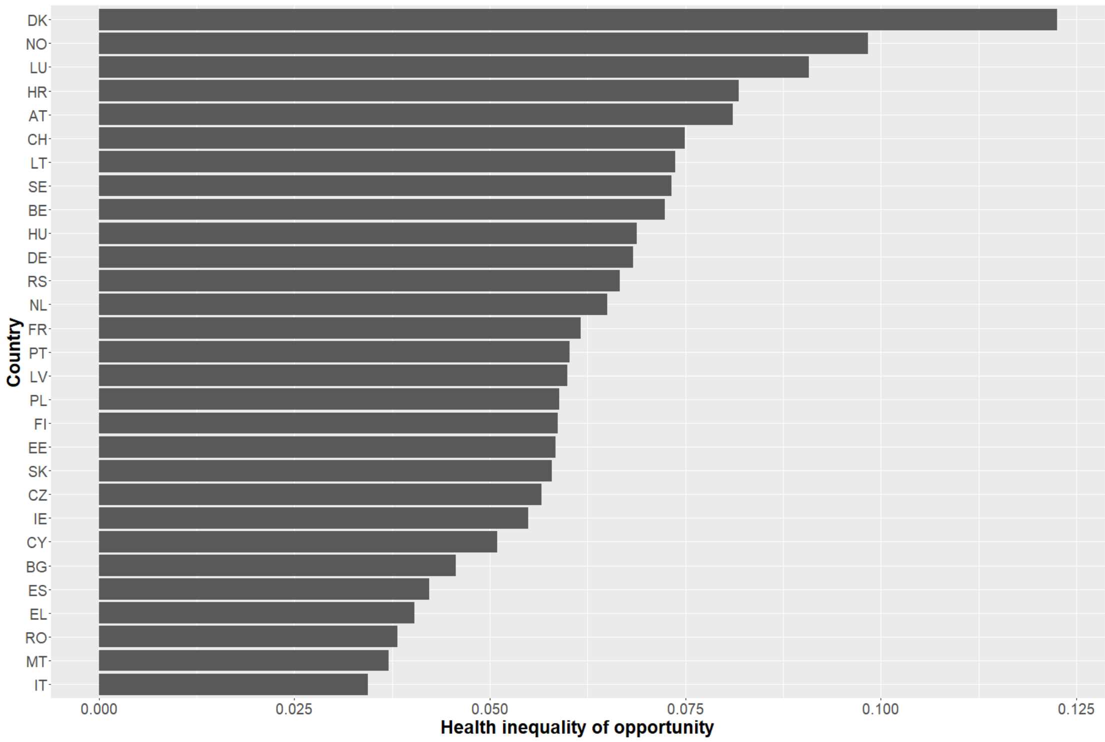
Overall, we see that when more outcomes are taken into account, countries that provide most equal opportunities for income do not necessarily do so for other dimensions of important life outcomes. Health and education are much more influenced by past conditions than income in Denmark, Norway and Luxembourg. As a consequence the difference in inequality of opportunity between them and other countries is reduced when all three outcomes are considered comparing to the situation when only income is. This best shows the value of multidimensional analyses. These results are consistent with Carneiro et al. (2021) and Heckman and Landersø (2022) who find that although in both Norway and Denmark income inequality of opportunity is low, there is a lot of educational inequality of opportunity. These countries achieve low income inequality of opportunity through *post-redistribution* (its tax and transfer programs), but not by *pre-redistribution* (i.e. enhancing the human capital of children across generations). Luxembourg is characterized by considerable educational inequalities (Hadjar et al. 2015, 2018) due to high immigrant rate of the population and language barriers that the system does not adequately address. Thus, equality of opportunity in well-being in those countries is not high - a conclusion that emerges only when one looks beyond income.

Figure 2: Estimates of $I^{AOEA}$ for income: full measure



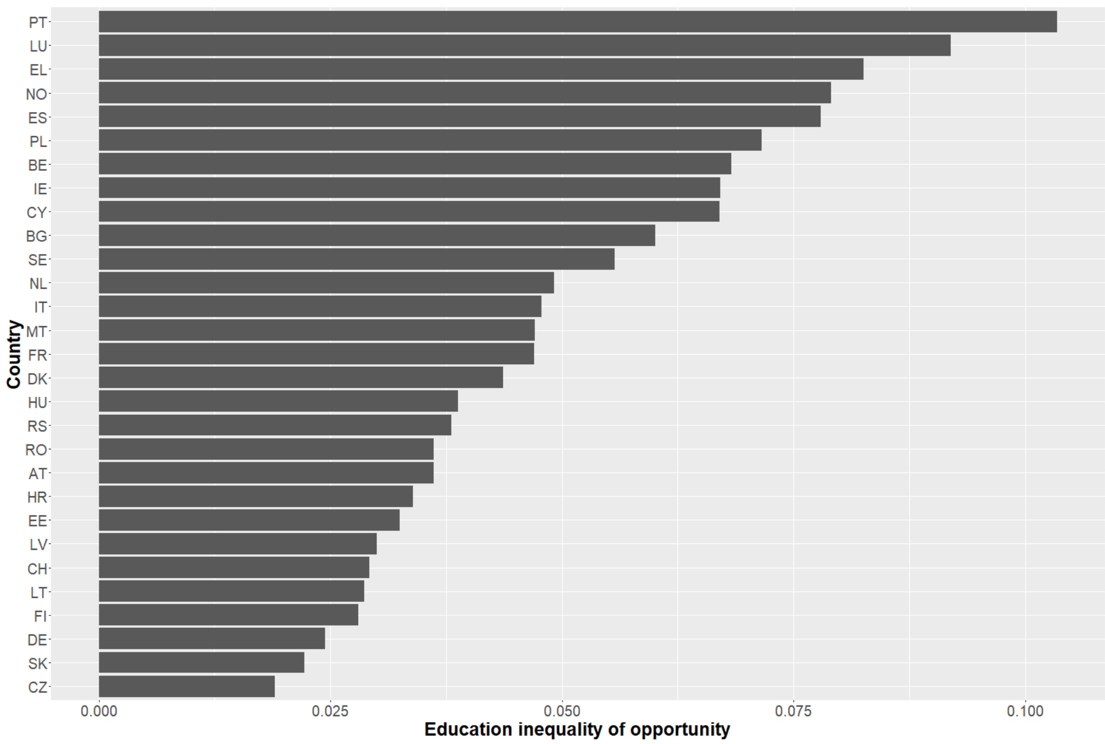

Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Figure 3: Estimates of $I^{AOEA}$ for health: full measure



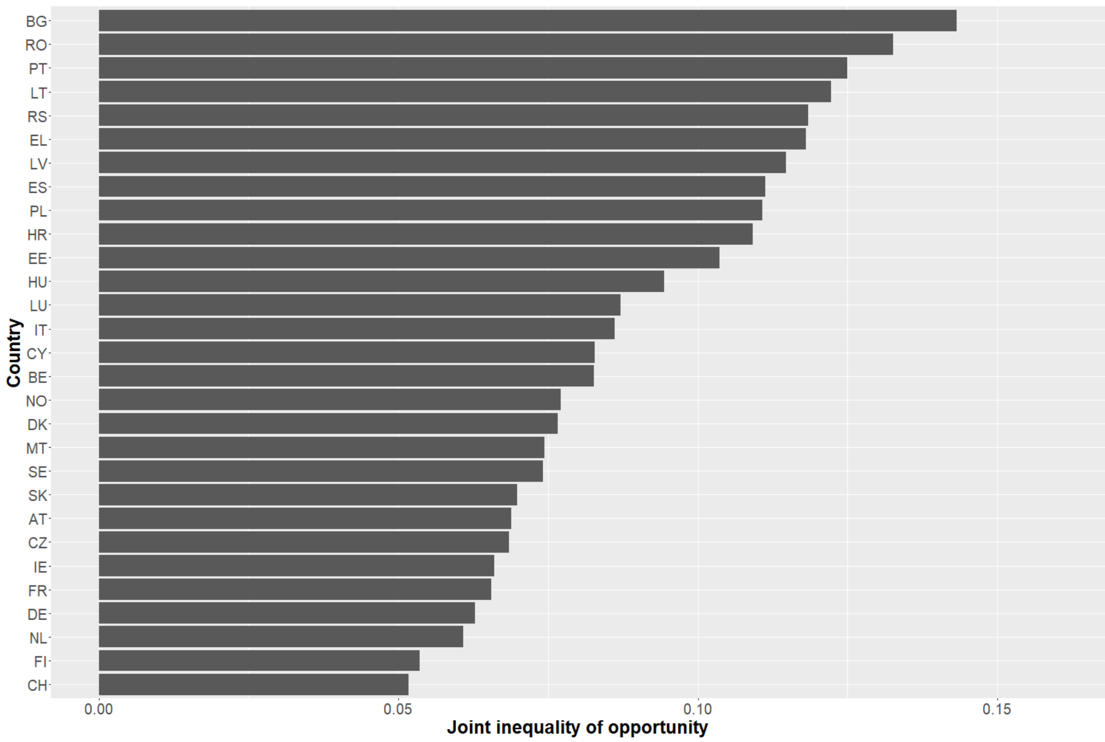

Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Figure 4: Estimates of $I^{AOEA}$ for educational attainment: full measure




Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

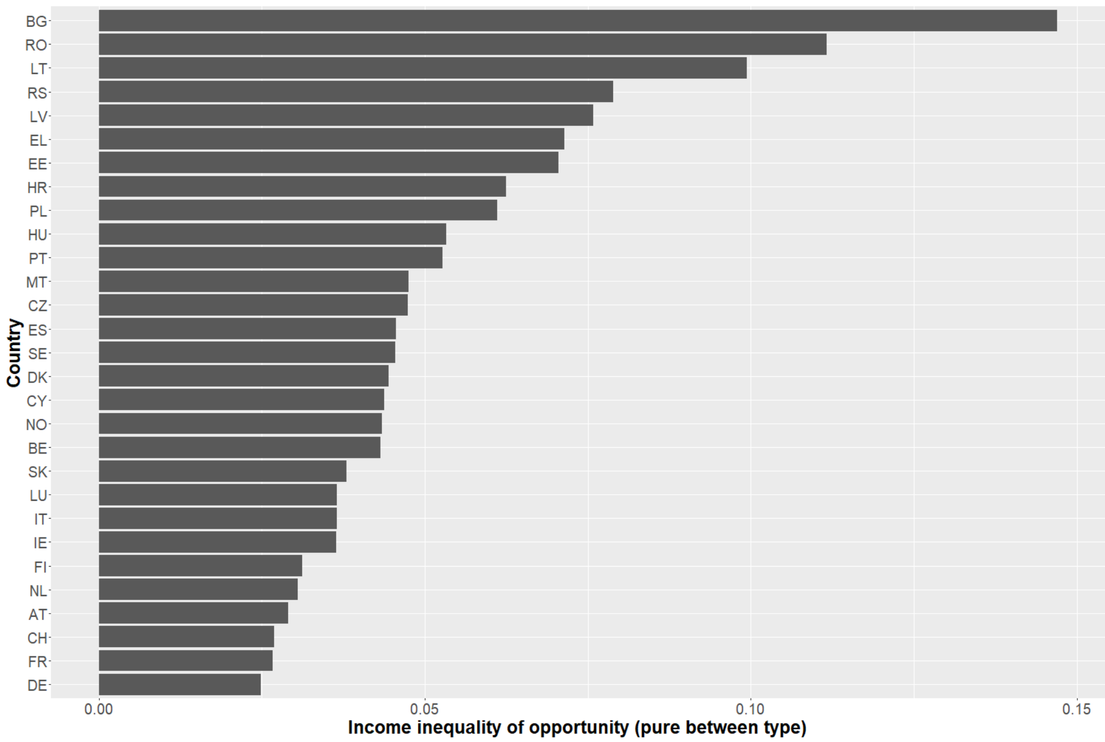
Figure 5: Estimates of $I^{AOEA}$ for joint outcomes: full measure




Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

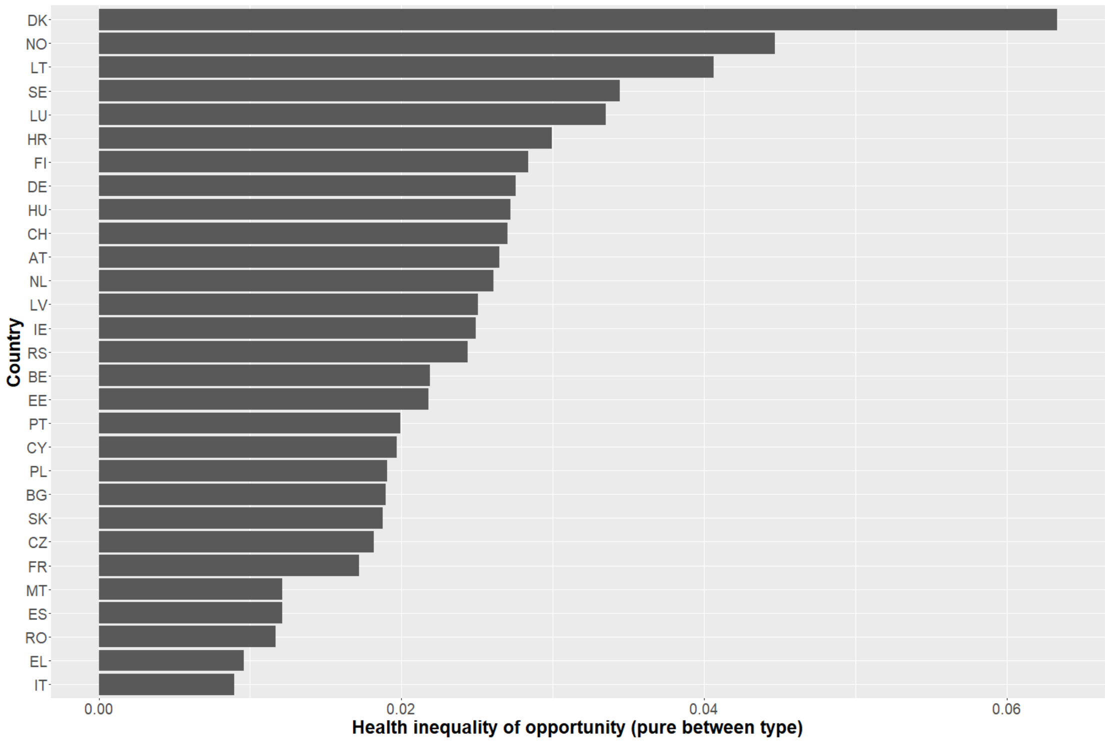
When no attention is paid to inequality within types, we have pure between type inequality (Figures 6 - 9). When compared to the full measure, the values of pure between type IOP comprise between 25% and 55% of full IOP. The rest of the variation comes from within type dispersion. Roughly speaking, countries with high/medium/low full IOP are countries with high/medium/low pure between type IOP, but they reshuffle within these broad groups. Correlations between full measure estimations and pure between type are high: 0.908 for joint, 0.904 for income, 0.922 for health and 0.960 for education. Bulgaria (0.147) has significantly higher income pure between type IOP than other countries, Denmark (0.063) stands out negatively for health and Luxembourg (0.054) and Portugal (0.052) for education, the latter being the same as in the full measure computation. Income is the most dominating dimension, therefore Bulgaria has much higher joint pure between type IOP (0.076) than others, but Portugal is also high on the list (0.045).

Figure 6: Estimates of $I^{AOEA}$ for income: pure between type



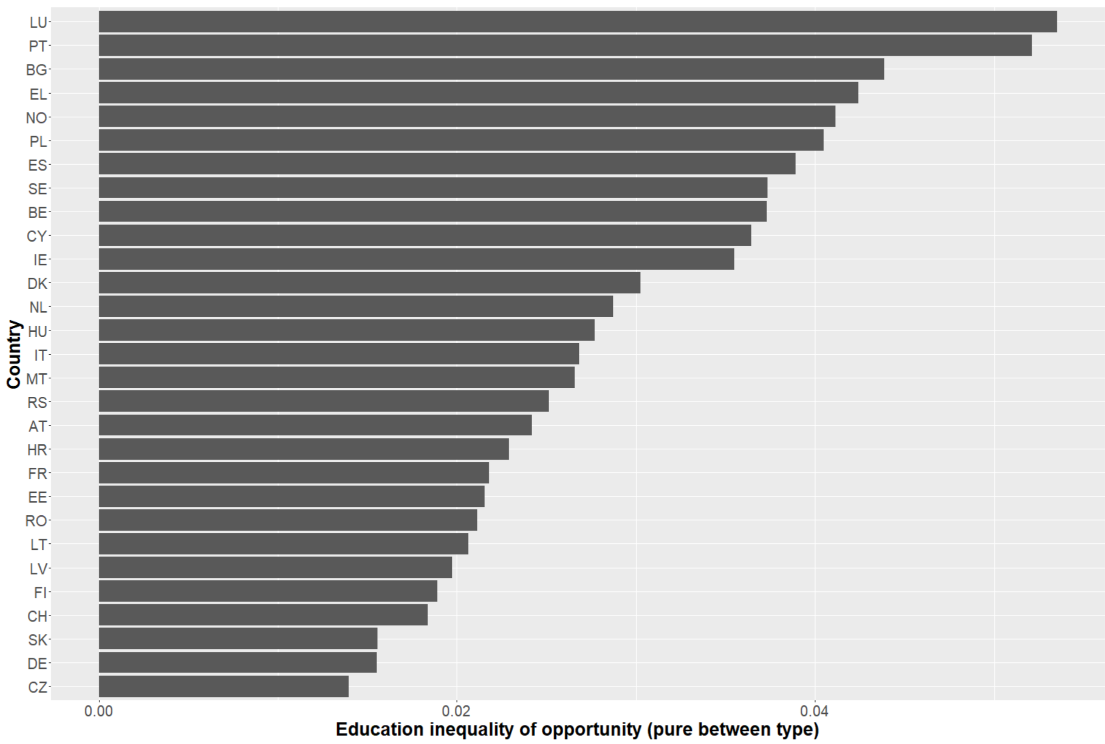

Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Figure 7: Estimates of $I^{AOEA}$ for health: pure between type



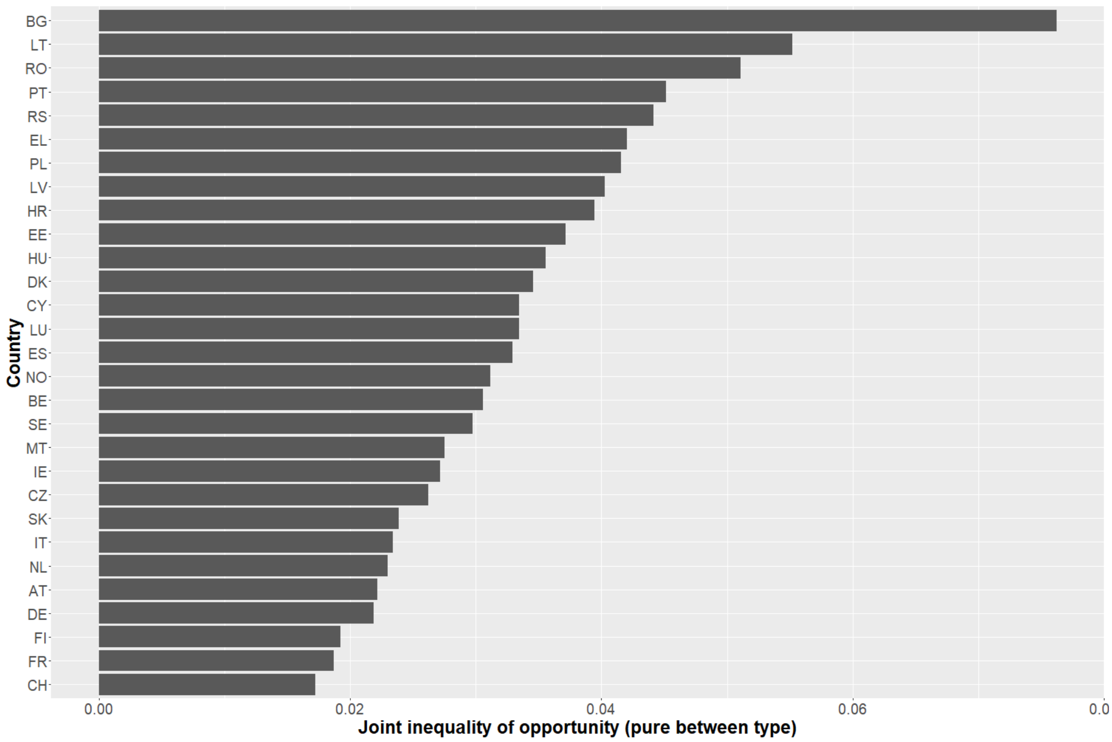

Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Figure 8: Estimates of $I^{AOEA}$ for educational attainment: pure between type




Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Figure 9: Estimates of $I^{AOEA}$ for joint outcomes: pure between type




Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

In order to better understand the similarities and differences in obtained estimates, we use K-means algorithm to group countries using four variables: joint IOP and three unidimensional IOPs. Figures 10-12 present the obtained groups in two-dimensional spaces: joint & income, joint & health, and joint & edu. The axes are normalized with respect to the worst (highest inequality) country on each axis. For example, in case of joint & income space, Bulgaria is the most unequal country in both. For joint & income space, countries are spread out almost along the diagonal confirming that income as the most unequal dimension drives joint IOP the most, but not a hundred percent, which our multidimensional analysis reveals. In general, as the algorithm shows, we can distinguish four groups.

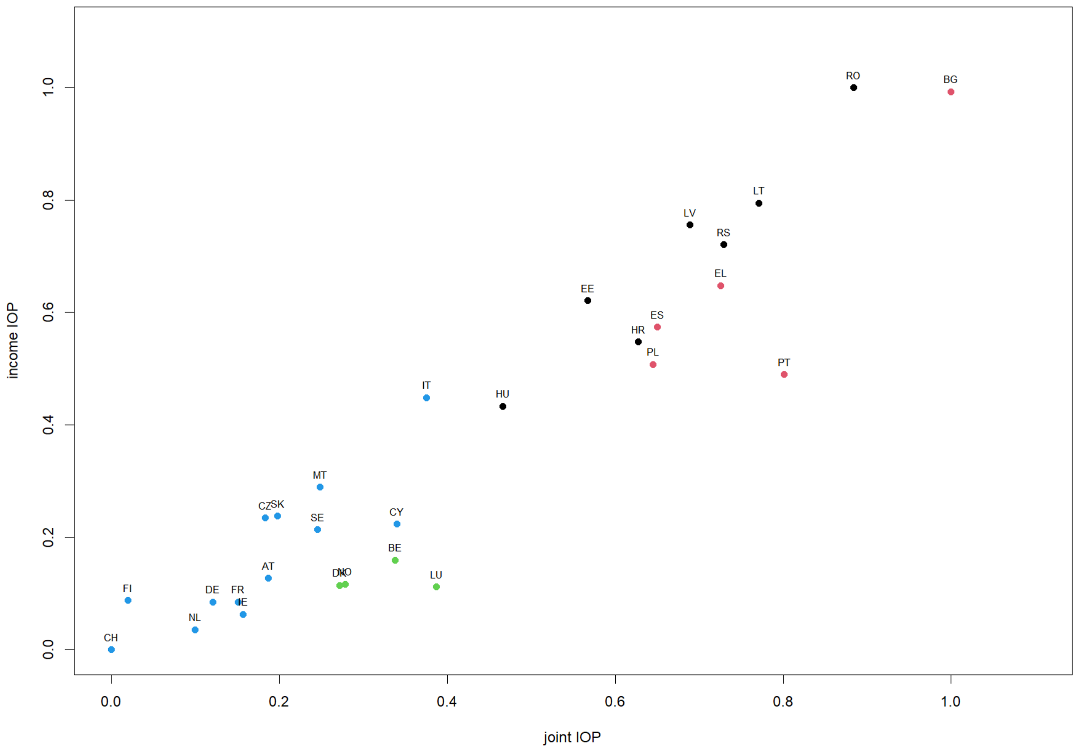
*Green* group is the largest. These are mostly countries in the vicinity of France and German, all German speaking countries, Slovak and Czech Republic, Sweden and Finland. For these countries income and joint IOPs increase jointly and they occupy from low levels of both variables (Switzerland) to medium levels (Italy, Hungary). This is the most equal group. When we look at health & joint graph, for this group health IOP is negatively correlated with joint IOP. Educational IOP seems to be relatively (bearing in mind the normalization of axes) higher than joint IOP for this group.

*Red* group (Denmark, Norway, Belgium Luxembourg, parts of Scandinavia and Benelux). These are particularly interesting countries from our multidimensional perspective; rich countries that have low income IOP, but fairly substantial joint IOP, which indicates that while these countries are equal with respect to income, they are not so with respect to other dimensions. They have substantial educational and health IOP.

*Blue* group (Poland and Southern Europe). This is a group of countries with quite high joint IOP and relatively high income IOP. This should be seen in comparison to the black group that lies on the opposite side of the diagonal i.e. has similar levels of joint IOP but higher income IOP. There is heterogeneity in this small group; Portugal has high joint IOP but not so high income IOP, which, as we already known - indicates that it is mostly education and to some extent health that drive inequality in Portugal. Poland too is slightly off the diagonal in income & joint space, meaning that although it has lower income IOP than, for example, Spain, it has similar joint IOP. And the reason we see on health & joint graph – Poland has higher health IOP than some other members of the group.

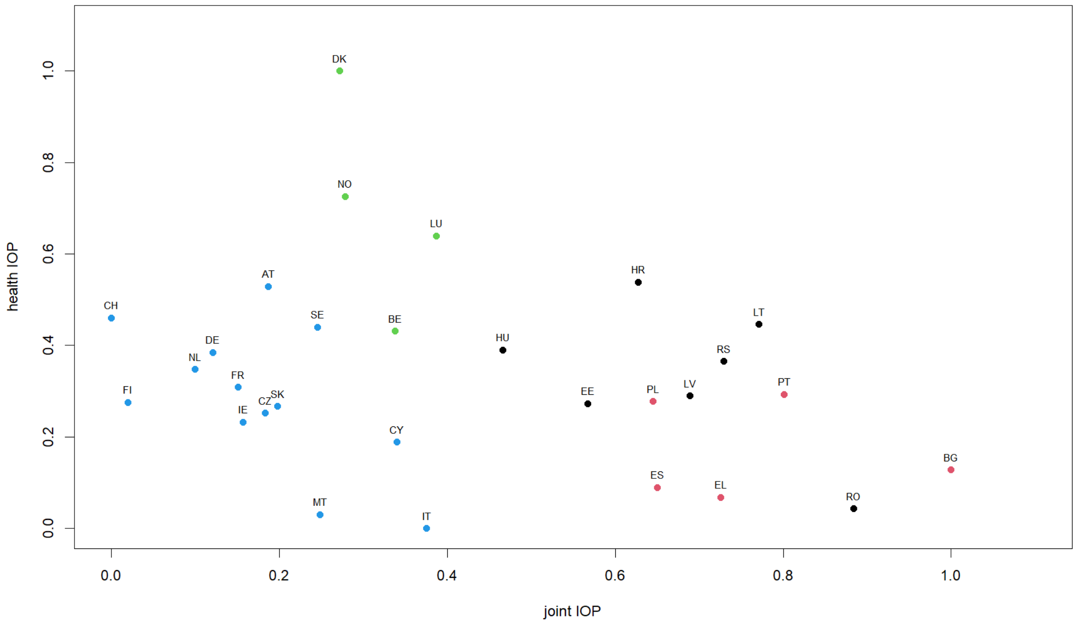
*Black* group (Baltics and Eastern Europe). This is a group of higher income & joint IOP, with income & joint increasing together and relatively high income IOP. These countries have similar levels of educational IOP and it is hard to find any pattern for health IOP in this group.

Figure 10: K-means grouping: joint-income space



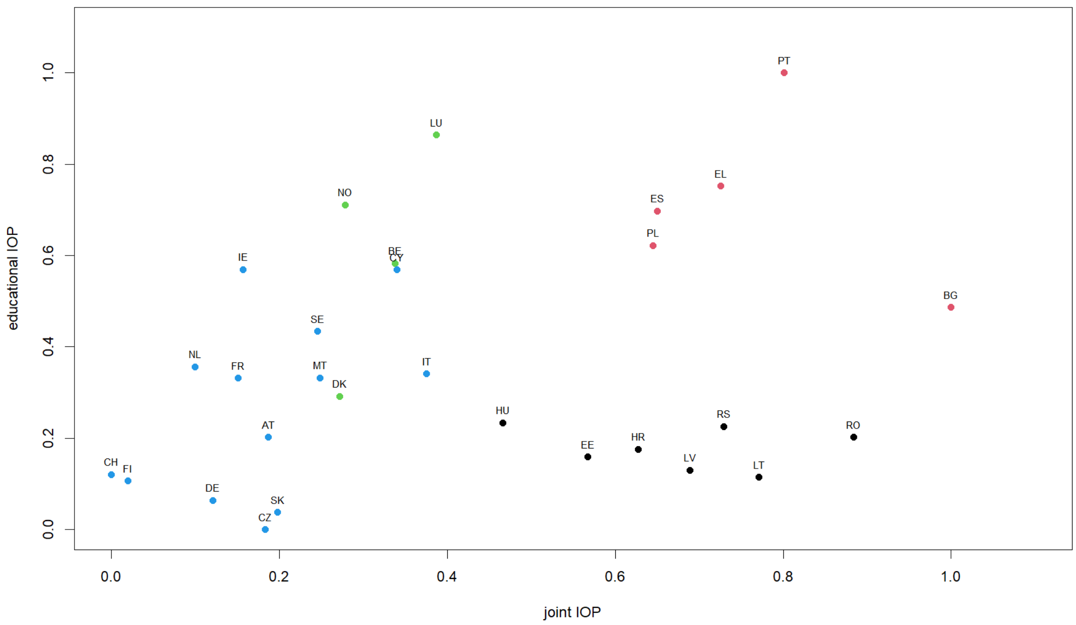

Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Figure 11: K-means grouping: joint-health space




Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

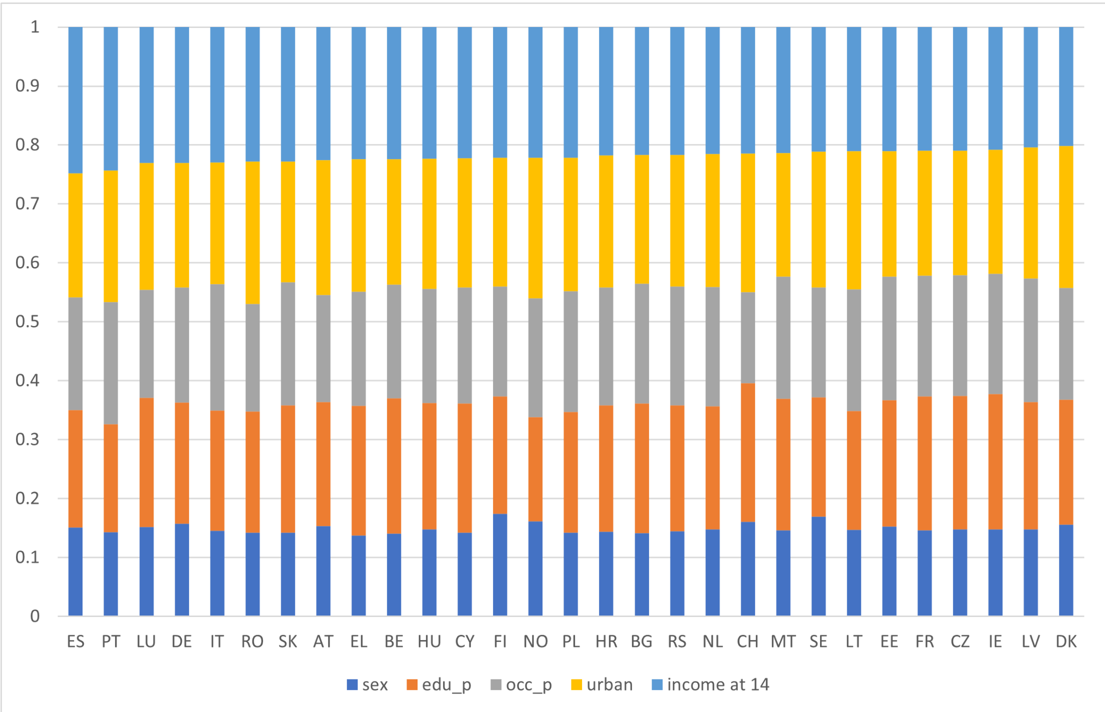
Figure 12: K-means grouping: joint-education space




Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

We perform Shapley value decomposition for each country to study the impact of each circumstance on the overall IOP. The contributions turn out to be fairly stable across European countries; the most varied contributions are for parental occupation. The values of contributions are also pretty balanced across circumstances. The lowest mean contribution is for sex (0.149), then for parental occupation (0.197), then education of parents (0.212), followed by household income at 14 years old (0.22) and area of residence (0.221). Finally, we run several robustness checks; the results are in the Appendix B. While the values of inequality may change, the results do not change qualitatively and countries' rankings are quite stable.

Figure 13: Shapley value decomposition for cicumstance variables




Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

# 6 Concluding remarks

In this paper, we develop a normative approach to the measurement of inequality of opportunity in a multidimensional setting. We obtain characterization theorems for two classess of welfare functions. The first class that fulfills ex-ante compensation and utilitarian reward is implemented via Generalized Lorenz Dominance applied to each attribute separately. The second class that obeys ex-ante compensation and inequality averse reward is implemented by the welfarist version of Lorenz Dominance, namely, Lorenz ordering of type-aggregate utilities. The class of utility functions used in the inequality averse scenario is submodular, hence it captures the dependence between attributes. Furthermore, we develop normative (Atkinson, 1970; Kolm 1969; Sen, 1973) inequality of opportunity indices for the two classes of welfare functions and study their properties. The key class of indices is sensitive to multidimensional inequality, both between and within types. Finally, we illustrate the methods developed with the application to multidimensional equality of opportunity in Europe. This reveals several significant differences between multidimensional EOP, which uses income, health and education as dimensions of interest, and income only EOP.

This paper should be considered as a starting point in the topic of extending EOP theory to a multidimensional setting. There are several issues that naturally come to mind when thinking about future research in this direction. The EOP literature is very diversified in its postulates, philosophical views and measurement approaches. The current paper focuses on the ex-ante approach. The most natural extension is to study the ex post approach in all its variants (see Roemer, 1998, Peragine, 2004, Fleurbaey 2008 and Fleurbaey et al. 2017). There are many other aspects (i.e. including luck component, norm-based measures) not considered in this paper that could be modified to a multidimensional setting. Another important extension is to relax the underlying assumption of fixed population distribution and allow for both changes in $X$ and the partitions of $N$ into types (see e.g., Peragine 2004). Many new topics for future research emerge when one goes deeper into the nature of multidimensional outcomes. In particular, in the multidimensional case some attributes may not be transferrable. For example, health which we use in this paper is considered to be such an attribute. However, transferrability is not the focus of this paper, but there are papers that focus specifically on this issue. It would be interesting to extend unidimensional EOP in this direction along the lines of Bosmans et al. (2009), Muller and Trannoy (2012) and Gravel and Moyes (2012) who propose different formulations of the transfer principle for the case in which only some attributes can be transferred.

## Acknowledgements

The study was funded by the National Science Centre in Poland (Grant No. 2016/23/G/HS4/04350). We are grateful to Paul Hufe and Radosław Kurek for their insightful comments. We thank the participants of The Eight ECINEQ Meeting in Paris and the participants of the "Opportunities, Mobility and Well-Being" workshop in Warsaw for insightful comments and discussion.

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# Appendix A

*Proof of Lemma 1.* By INWT we know that $W(X) = W(Y)$ where $X$ is PDT of $Y$. We show that $X_\mu$ can be obtained as a result of a sequence of PDT transformations of $X$. It suffices to show it for some arbitrary type $h$.

We denote by $Y^{l,p}, l \geq p$ the matrix obtained after $l$ PDT transformations of $X^h$ where a transfer takes place between $p$-th and other individual. In the first step, $p = 1$, we perform a sequence of transfers after which the first individual ends up getting type-means on all dimensions. Let $Y^{1,1} = X^h$. In each step we make a PDT transfer between 1st individual and individual $i$-th such that the transferred amount is $\varepsilon_i = (\frac{i-1}{i}, \ldots, \frac{i-1}{i})$, i.e. $Y^{i,1}_{1j} = \frac{i-1}{i} Y^{i-1,1}_{1j} + \frac{1}{i} Y^{i-1,1}_{ij}$ and $Y^{i,1}_{ij} = \frac{i-1}{i} Y^{i-1,1}_{ij} + \frac{1}{i} Y^{i-1,1}_{1j}$. Finally, we have that $Y^{n,1}_{1j} = \sum \frac{1}{n} X^h_{ij} = (X_\mu)^h_{ij}$. We repeat the same procedure for other individuals. For an arbitrary $p$-th individual we have $Y^{1,p} = Y^{n,p-1}$ and $Y^{i,p}_{pj} = \frac{i-1}{i} Y^{i-1,p}_{pj} + \frac{1}{i} Y^{i-1,p}_{ij}$ and $Y^{i,p}_{ij} = \frac{i-1}{i} Y^{i-1,p}_{ij} + \frac{1}{i} Y^{i-1,p}_{pj}$. □

*Proof of Theorem 1.* By ADD we have $W(X) = \sum_{h=1}^{n} \sum_{i=1}^{N_h} U^h(X^h_{i.})$. We now show that $U^h(X^h_{i.}) = \sum_{j=1}^{k} U^h_j(X^h_{ij})$, $dU^h/dX > 0$ and $D^2 U^h$ is zero matrix. Let $X^h_{p.}, X^h_{q.}$ be two outcomes. From INWT and ADD we have that $U^h(X^h_{p.}) + U^h(X^h_{q.}) = U^h(X^h_{p.} + \varepsilon(X^h_{q.} - X^h_{p.})) + U^h(X^h_{q.} - \varepsilon(X^h_{q.} - X^h_{p.}))$. This is equivalent to $\frac{U^h(X^h_{p.}) - U^h(X^h_{p.} + \varepsilon \cdot (X^h_{q.} - X^h_{p.}))}{|\varepsilon \cdot (X^h_{q.} - X^h_{p.})|} = \frac{U^h(X^h_{q.} - \varepsilon \cdot (X^h_{q.} - X^h_{p.})) - U^h(X^h_{q.})}{|\varepsilon \cdot (X^h_{q.} - X^h_{p.})|}$. Moving to zero in the limit with the norm of $\varepsilon$ we obtain equality of directional derivatives for arbitrary points $X^h_{p.}, X^h_{q.}$, namely $D_v U^h(X^h_{p.}) = D_v U^h(X^h_{q.})$, where $v = \frac{X^h_{q.} - X^h_{p.}}{|X^h_{q.} - X^h_{p.}|}$. By taking $X^h_{p.}, X^h_{q.}$ such that $v$ are base vectors we get $\frac{\partial}{\partial x_j} U^h(X^h_{p.}) = \frac{\partial}{\partial x_j} U^h(X^h_{q.})$ and in consequence

$\frac{\partial^2}{\partial x_j^2} U^h(X_{p.}^h) = 0$ for $1 \leq j \leq k$. We get that $U^h$ is of the form

$$U^h(X_{p.}^h) = b + \sum_{j=1}^{k} a_j X_{pj}^h + \sum_{j=1}^{k} \sum_{m=j}^{k} a_{jm} X_{pj}^h X_{pm}^h + \cdots + a_{12\ldots k} \Pi_{j=1}^k X_{pj}^h$$

and we need to show that coefficients other than $b$ and $a_j$ are zero. We will do this for $k = 3$ and for $k > 3$ it runs along the same lines. Let us take $v = \frac{\sqrt{3}}{3}(1, 1, 1)$, then $D_v U^h(X_{p.}^h) = \frac{\sqrt{3}}{3}(\frac{\partial}{\partial x_1} U^h(X_{p.}^h) + \frac{\partial}{\partial x_2} U^h(X_{p.}^h) + \frac{\partial}{\partial x_3} U^h(X_{p.}^h)) = \frac{\sqrt{3}}{3}(a_1 + a_{12} X_{p2}^h + a_{13} X_{p3}^h + a_{123} X_{p2}^h X_{p3}^h + a_2 + a_{23} X_{p3}^h + a_{12} X_{p1}^h + a_{123} X_{p1}^h X_{p3}^h + a_3 + a_{23} X_{p2}^h + a_{13} X_{p1}^h + a_{123} X_{p1}^h X_{p2}^h)$. From $X_{q.}^h = X_{q.}^h + cv$ and $D_v U^h(X_{q.}^h) - D_v U^h(X_{p.}^h) = 0$ we get $2c(a_{12} + a_{23} + a_{13}) + a_{123}(3c^2 + 2c(X_{p1}^h + X_{p2}^h + X_{p3}^h)) = 0$. From arbitrariness of $X_{p.}^h$ and $c$ we get $a_{12} + a_{23} + a_{13} = a_{123} = 0$. Now let us take $v = \frac{\sqrt{2}}{2}(1, 1, 0)$ , then $D_v U^h(X_{p.}^h) = \frac{\sqrt{2}}{2}(\frac{\partial}{\partial x_1} U^h(X_{p.}^h) + \frac{\partial}{\partial x_2} U^h(X_{p.}^h)) = \frac{\sqrt{2}}{2}(a_1 + a_{12} X_{p2}^h + a_{13} X_{p3}^h + a_2 + a_{23} X_{p3}^h + a_{12} X_{p1}^h)$. From $X_{q.}^h = X_{q.}^h + cv$ and $D_v U^h(X_{q.}^h) - D_v U^h(X_{p.}^h) = 0$ we get $c(2a_{12} + a_{23} + a_{13}) = ca_{12} = 0$, so from arbitrariness of $c$, $a_{12} = 0$. Analogously, $a_{23} = a_{13} = 0$, so

$$U^h(X_{p.}^h) = b + \sum_{j=1}^{k} a_j X_{pj}^h$$

Thus we obtain that $W$ is of the form

$$W(X) = Nb + \sum_{h=1}^{n} \sum_{i=1}^{N_h} \sum_{j=1}^{k} a_j^h X_{ij}^h.$$

In the reverse implication, this functional form of $W$ clearly satisfies ADD and MON and INWT.

Now to show that $dU^s/dX > dU^r/dX \geq 0$ for $s < r$, we use IABT. Let $X$ be obtained from $Y$ by PDTT between types $r, s$ and individuals $p, q$ from these types, respectively. Let us take $X_{p.}^r, X_{q.}^s$ equal in all but one attribute. Then by IABT we have

$$W(X) > W(Y) \iff U^r(X_{pj}^r) + U^s(X_{qj}^s) > U^r(Y_{pj}^r) + U^s(Y_{qj}^s).$$

$$\iff U^r(Y_{pj}^r(1 - \varepsilon) + Y_{qj}^s \varepsilon) + U^s(Y_{qj}^s(1 - \varepsilon) + Y_{pj}^r \varepsilon) > U^r(Y_{pj}^r) + U^s(Y_{qj}^s) \iff$$

$$\iff U^s(Y_{qj}^s + \varepsilon(Y_{pj}^r - Y_{qj}^s)) - U^s(Y_{qj}^s) > U^r(Y_{pj}^r) - U^r(Y_{pj}^r - \varepsilon(Y_{pj}^r - Y_{qj}^s)).$$

Division by $\varepsilon(Y_{pj}^r - Y_{qj}^s)$ with $\varepsilon \to 0$ gives us that $dU^s/dX \geq dU^r/dX$ for $\theta_s \prec \theta_r$. For linear function this implies restriction on coefficients $a_j^h > a_j^{h+1}$ for all $j$ in (7). For such $W$ the rest of the proof follows from Theorem 1 (Peragine 2004) for any $j$.

□

*Proof of Theorem 2.* We start by showing that $W \in \mathcal{W}^{AOEA}$ if and only if it is of the form

$$W(X) = \sum_{h=1}^{n} \sum_{i=1}^{N_h} U^h(X_{i.}^h)$$

with $U^h$ being increasing, concave, type-concave and submodular. Obviously, ADD is equivalent to $W(X) = \sum_{h=1}^{n} \sum_{i=1}^{N_h} U^h(X_{i.}^h)$ and MON is equivalent to $dU^h/dX \geq 0$. In the proof of Theorem 1 we prove that IABT is equivalent to $dU^s/dX \geq dU^r/dX$, $\theta_s \prec \theta_r$ for general function $U^h$, so it applies here too. Now, let $Y$ be a CIT of $X$. Then, from IAWT we have $W(Y) \leq W(X)$, from which via ADD we further obtain

$$U^h(X_{p.}^h \wedge X_{q.}^h) + U^h(X_{p.}^h \vee X_{q.}^h) \leq U^h(X_{p.}^h) + U^h(X_{q.}^h),$$

which is the definition of submodularity. Now let $X$ be obtained from $Y$ via PDT with $\varepsilon = 0.5$. Then, $X_{pj}^h = X_{qj}^h = \frac{Y_{pj}^h + Y_{qj}^h}{2}$ and $U^h\left(\frac{Y_{pj}^h + Y_{qj}^h}{2}\right) \geq \frac{U^h(Y_{pj}^h) + U^h(Y_{qj}^h)}{2}$. Thus $U^h$ is concave. Now, finishing the proof of the first implication, let us assume that $X \succeq_{LD(\mathcal{U}^{AOEA})} Y$. Then $\sum_{h=1}^{l} u_h^X \geq \sum_{h=1}^{l} u_h^Y \quad \forall_{l=1,\ldots,n}$. Taking $l = n$ we get

$$\sum_{h=1}^{n} u_h^X \geq \sum_{h=1}^{n} u_h^Y \iff \sum_{h=1}^{n} \sum_{i=1}^{N_h} U^h(X_{i.}^h) \geq \sum_{h=1}^{n} \sum_{i=1}^{N_h} U^h(Y_{i.}^h) \iff W(X) \geq W(Y).$$

We now turn to the second implication. We have $W(X) \geq W(Y)$ for any $W \in \mathcal{W}^{AEOA}$, and we want to prove that $X \succeq_{LD(\mathcal{U}^{AOEA})} Y$. In other words, we have $\sum_{h=1}^{n} u_h^X \geq \sum_{h=1}^{n} u_h^Y$ and want to prove that this implies $\sum_{h=1}^{l} u_h^X \geq \sum_{h=1}^{l} u_h^Y$ for any $l \in \{1, \ldots, n\}$. We proceed by contradiction. Let us assume that for some $l \in \{1, \ldots, n\}$, $\sum_{h=1}^{l} u_h^X < \sum_{h=1}^{l} u_h^Y$. For $l = n$, $W(X) < W(Y)$, which is a contradiction. For $l < n$, let us denote $A = \sum_{h=1}^{l} u_h^Y - \sum_{h=1}^{l} u_h^X > 0$. Define $W'$ with $U'^h = U^h$ for $h \leq l$ and $U'^h = \frac{A}{2(\sum_{h=l+1}^{n} u_h^X - \sum_{h=l+1}^{n} u_h^Y)} U^h$, where $\frac{A}{2(\sum_{h=l+1}^{n} u_h^X - \sum_{h=l+1}^{n} u_h^Y)} < 1$, so that $W' \in \mathcal{W}^{AEOA}$. Since $\sum_{h=1}^{n} u_h^X - \sum_{h=1}^{n} u_h^Y = \frac{A}{2} - A < 0$, which contradicts the assumption that $\sum_{h=1}^{n} u_h^X \geq \sum_{h=1}^{n} u_h^Y$. □

*Proof of Theorem 3.* First we show that $I$ is an inequality measure. It is continuous since $W$ is. For $X^\mu$, $I(X^\mu) = 1 - \frac{W(X^\mu)}{W(X^\mu)} = 0$. Finally, since PDTT does not change outcomes' total sums, we get that $Y^\mu = X^\mu$, so $W(\delta(Y)Y^\mu) \leq W(\delta(X)X^\mu)$ implies $\delta(Y) \leq \delta(X)$. From Definition 1, we have $I(X) = 1 - \delta(X)$ where $\delta(X)$ satisfies $W(\delta(X)X^\mu) = W(X)$. Since by Theorem 1 any $W \in \mathcal{W}^{AOEN}$ is linear, we have $W(\delta(X)X^\mu) = \delta(X)W(X^\mu) = W(X) \iff \delta(X) = \frac{W(X)}{W(X^\mu)}$, proving 1. Now by Lemma 1 we obtain 2.

□

*Proof of Theorem 4.* First, we prove that $I$ is an inequality measure. Continuity is obvious. $I(X^\mu) = 0$, since for $X^\mu$, we have $W(X^\mu) = W(\delta(X^\mu)X^\mu)$, so $\delta(X^\mu) = 1$. Furthermore, since PDTT does not change outcomes' total sums, we have $Y^\mu = X^\mu$. Then $W(Y) \leq W(X)$ implies $\delta(Y) \leq \delta(X)$. Relativity follows from the fact that $W(X) = W(\delta X^\mu) \iff W(XC) = W(\delta X^\mu C)$.

Let us fix $h$. We now use Theorem 1 in Tsui (1995) to obtain the expression for $\delta_h$ i.e. equality index for a single type $h$, defined as $W(X^h) = W(\delta_h X^h_\mu)$. For a given $h$ our axioms correspond to the axioms in Tsui (1995).[13] In such a case, Tsui (1995) shows that $W$ is equivalent to the sum of utility functions, where utility function is one of two forms: $U^h(X^h_i) = a_h \prod_{j=1}^k \left(X^h_{ij}\right)^{r_j}$ with $a_h < 0, r_j < 0$ or $\sum_{j=1}^k r_j \log\left(X^k_{ij}\right)$. Since the latter form does not satisfy IABT, point 2 follows automatically. For the former class of utility functions, the obtained equality index for a single type is $\delta_h(X^h) = \left[\frac{1}{N_h}\sum_{i=1}^{N_h}\prod_{j=1}^k\left(\frac{X_{ij}}{(X^h_\mu)_j}\right)^{r_j}\right]^{\frac{1}{\sum_{j=1}^k r_j}}$.

Now, relaxing the assumption of a single type, by ADD, we have $W(X^1, \ldots, X^n) = W(\delta_1 X^1_\mu, \ldots, \delta_n X^n_\mu)$ with $\delta_h$ as above. We write

$$W(\delta X^\mu) = W(X) \iff -\sum_{h=1}^n N_h a_h \prod_{j=1}^k \left(\delta(X^\mu)^h_{1j}\right)^{r_j} = -\sum_{h=1}^n N_h a_h \prod_{j=1}^k \left(\delta_h (X_\mu)^h_{1j}\right)^{r_j}$$

$$\iff \delta^{\sum_{j=1}^k r_j} \sum_{h=1}^n N_h a_h \prod_{j=1}^k \left((X^\mu)^h_{1j}\right)^{r_j} = \sum_{h=1}^n N_h a_h \delta_h^{\sum_{j=1}^k r_j} \prod_{j=1}^k \left((X_\mu)^h_{1j}\right)^{r_j} \iff$$

$$\iff \delta = \left(\sum_{h=1}^n w_h \frac{U^h\left((X_\mu)^h_1\right)}{U^h\left((X^\mu)^h_1\right)}\right)^{\frac{1}{\sum_{j=1}^k r_j}},$$

from which we get (2). Further modifications lead to the form (3) of the index

$$I(X) = 1 - \delta(X) = I(X) = 1 - \left(\sum_{h=1}^n w_h \frac{U^h\left((X_\mu)^h_1\right)}{U^h\left((X^\mu)^h_1\right)}\right)^{\frac{1}{\sum_{j=1}^k r_j}}$$

$$= 1 - \left(\sum_{h=1}^n w_h \prod_{j=1}^k \left(\frac{(X_\mu)^h_{1j}}{(X^\mu)^h_{1j}}\right)^{r_j}\right)^{\frac{1}{\sum_{j=1}^k r_j}} = 1 - \left(\sum_{h=1}^n \frac{\delta_h(X) N_h a_h}{\sum_{h=1}^n N_h a_h} \prod_{j=1}^k \left(\frac{(X_\mu)^h_{1j}}{(X^\mu)^h_{1j}}\right)^{r_j}\right)^{\frac{1}{\sum_{j=1}^k r_j}}$$

$$= 1 - \left(\sum_{h=1}^n \frac{\left[\frac{1}{N_h}\sum_{i=1}^{N_h}\prod_{j=1}^k \left(\frac{X^h_{ij}}{(X_\mu)^h_{1j}}\right)^{r_j}\right] N_h a_h}{\sum_{h=1}^n N_h a_h} \prod_{j=1}^k \left(\frac{(X_\mu)^h_{1j}}{(X^\mu)^h_{1j}}\right)^{r_j}\right)^{\frac{1}{\sum_{j=1}^k r_j}}$$

[13] The difference would be for symmetry, which in our case works within type, but for now a type is fixed.

$$= 1 - \left( \sum_{h=1}^{n} \frac{a_h \sum_{i=1}^{N_h} \prod_{j=1}^{k} \left( \frac{X_{ij}^h}{(X_\mu)_{1j}^h} \right)^{r_j} \prod_{j=1}^{k} \left( \frac{(X_\mu)_{1j}^h}{(X^\mu)_{1j}^h} \right)^{r_j}}{\sum_{h=1}^{n} N_h a_h} \right)^{\frac{1}{\sum_{j=1}^{k} r_j}}$$

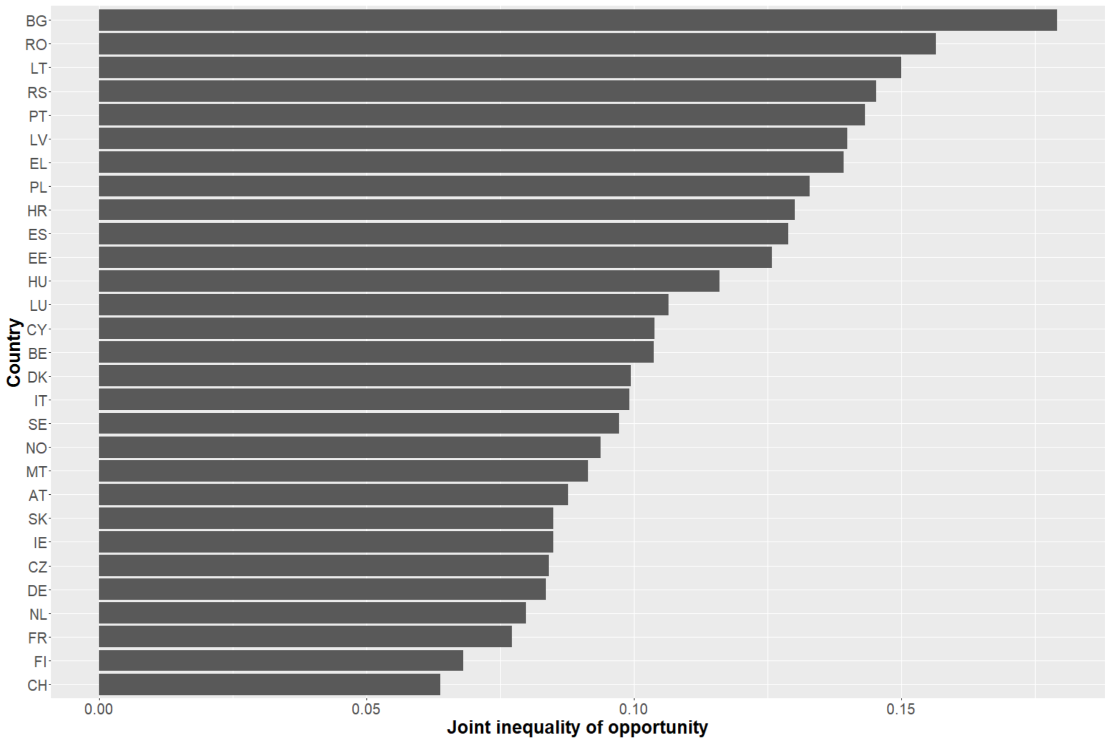
$$= 1 - \left( \sum_{h=1}^{n} \frac{a_h \sum_{i=1}^{N_h} \prod_{j=1}^{k} \left( \frac{X_{ij}^h}{(X^\mu)_{1j}^h} \right)^{r_j}}{\sum_{h=1}^{n} N_h a_h} \right)^{\frac{1}{\sum_{j=1}^{k} r_j}} .$$

Finally, when $\delta_h(X_\mu) = 1$ for all $h$, (4) and (5) follow easily from (2). □

# Appendix B

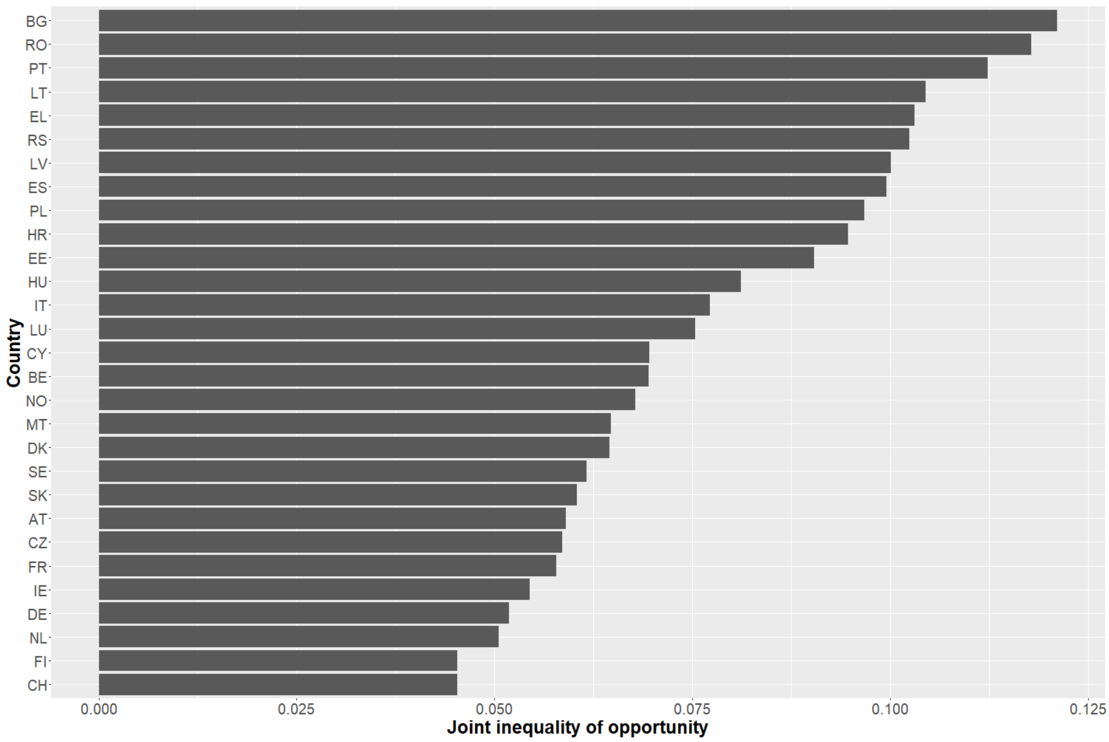
We run several robustness checks. Firstly, we vary type weights $a_h$. The weights used in the main computations are linear, here we use convex weights (Figure 14) and concave weights (Figure 15). Of course, the former give higher inequality scores and the latter give lower inequality scores, but the differences are not large and what is more important, rankings remain almost the same. Rank correlations with the joint IOP ranking with linear weights are as high as 0.98 for concave weights and 0.99 for convex weights. Next, we run our analysis deleting the types that have low number of observations (under 50). This almost does not change our base results, neither quantitatively or in terms of the ranking (Figure 16). Finally, we vary dimension weights, first giving higher inequality aversion $r_j = -0.8$ to all dimensions (Figure 17 and then assigning lower weight $r_j = -0.2$ the most unequal dimension, namely income (Figure 18). For the former check, nothing changes except for the inequality going up as expected. For the latter check, when lower weight is given to income, Portugal emerges as the most unequal country. This is expected; as mentioned, Portugal is characterized by high educational inequality of opportunity and medium levels of other types of inequality.

Figure 14: Quadratic type weights: joint IOP



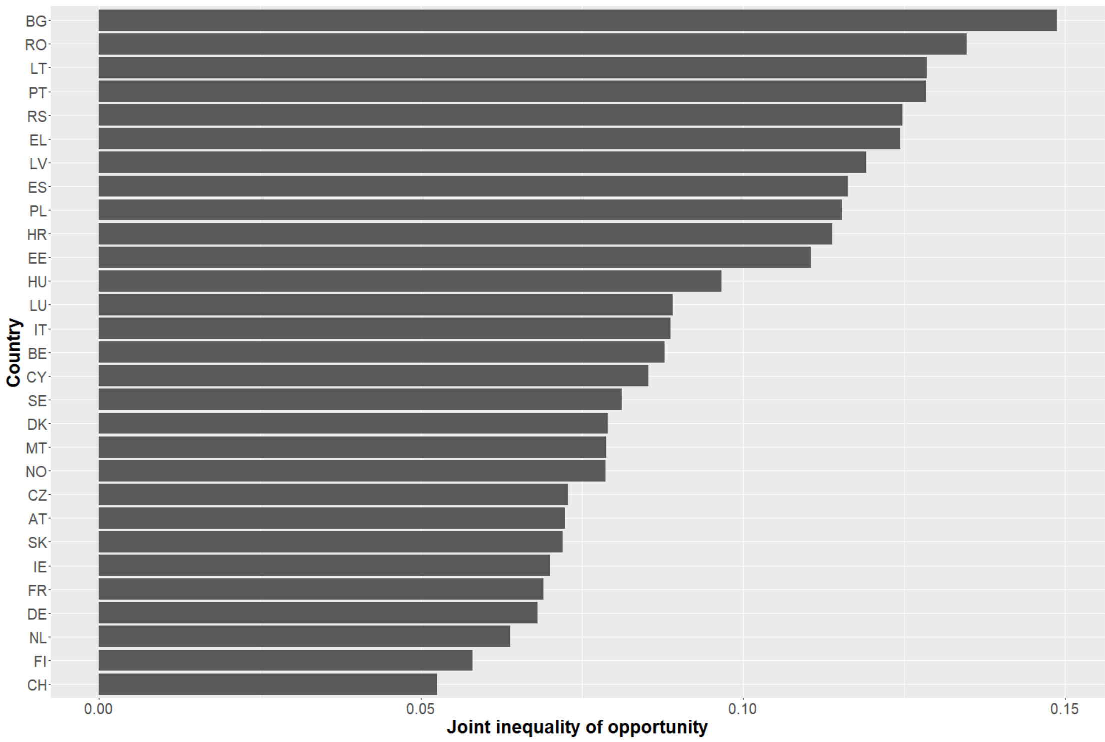

Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Figure 15: Square root type weights: joint IOP



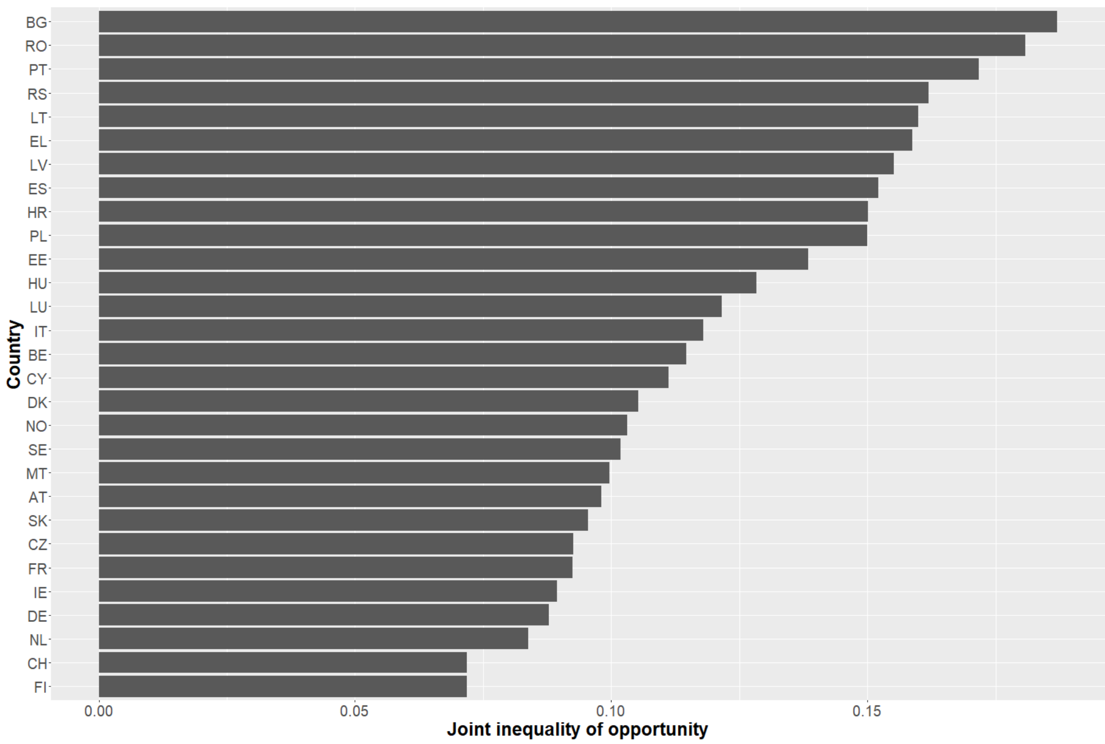

Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Figure 16: Deleting types under 50 observations: joint IOP



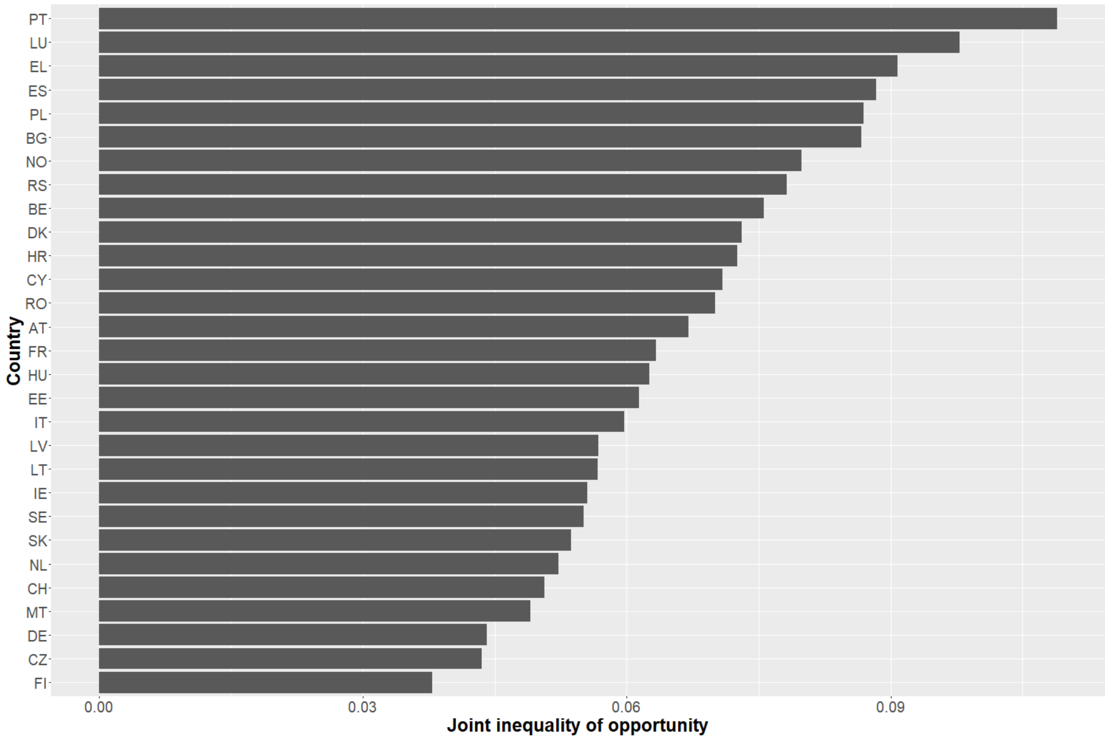

Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Figure 17: Dimension weights -0.8 for all dimensions: joint IOP




Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.

Figure 18: Dimension weights -0.2 for income and -0.8 for the rest: joint IOP




Note: Data come from EU Statistics on Income and Living Conditions Wave 2019.