The formation of the taste uncertain preferences

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Abstract

This paper provides a language to describe the formation of preferences under taste uncertainty. I propose to consider the taste uncertain consumer as equipped with a probabilistic measure on the space of all permissible preference relations. This measure captures the beliefs of the consumer regarding their real preference between alternatives and serves as a common basis for context dependent choice procedures. I construct a family of probabilistic measures that fulfill this role and use this construction to define preference relations that capture the expected preferences of the consumer, their perception of risk associated with a given choice that originates from their belief uncertainty and of their expected value of information provided by consumption. Based on the properties of those relations, I provide a theoretical justification for the link between taste uncertainty and the preference reversal paradox.

Keywords: Taste uncertainty, Preference discovery, Preference reversal, Learning through consumption, Preference formation.

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1 Introduction

Alice is on vacation in Thailand. She goes to a restaurant, browses through the menu and thinks what to order. Economic theory postulates that Alice should come to the restaurant equipped with some preference relation, and pick the element from the menu that is maximal with respect to it. However, in the menu Alice finds a dish named "durian risotto" and she has never eaten durian before. I may reasonably expect, that in this situation Alice cannot state with any real certainty what is her preference ranking of all the alternatives in the menu. In other words, she does not know her own preferences.

Taste uncertainty is not a new topic. It can be found in economic literature at least since the contribution of Kreps (1979). He showed that weak axioms on preferences are sufficient to represent introspective uncertainty using a subjective set of states. This has been achieved by the additional assumption of a preference for flexibility, which states that the consumer should prefer a menu that is larger with respect to inclusion. One potential justification for this axiom is the existence of taste shocks that might affect future preferences, which then makes a larger menu a safer choice. Piermont et al. (2016) and Cooke (2017) provide two recent extensions of this model that condition learning of the subjective state space on consumption.

Both Piermont et al. (2016) and Cooke (2017) assume, that the consumer facing uncertainty about their real taste comes equipped with some interim preference relation. This relation is only loosely connected to the real preferences of the consumer, that is revealed by consumption. From my perspective, the main problem with this approach is that it is not really clear what those interim preferences actually represent.

Let us again consider the case of Alice in the restaurant. As noted before, without taste uncertainty it would be sufficient to choose from the menu the maximal element with respect to her preference relation. However, under taste uncertainty it is unclear what exactly such a preference relation represents, as there are many factors in play. Firstly, she might experiment, meaning that she might choose to try durian even if she suspects it won't be any good, just to discover her preferences and avoid future uncertainty. Alternatively, she might focus on what she believes will bring her the highest immediate utility from consumption. However in this case, the question remains how does she evaluate durian as an option? Let us assume that from what she has heard, Alice believes she might like it. However, when compared with fish and chips, it is a risky choice, as Alice has already tried fish and chips in this restaurant and knows what she gets — whereas durian might turn out to be great, terrible or anything in between.

Psychological literature, most notably Tversky et al. (1988), suggests that the consumer choice is procedure dependent and which procedure is applied depends on the circumstances in which the decision is made. It certainly seems reasonable to expect it to be the case when taste uncertainty is present. For example, how long Alice plans to stay in Thailand might impact on whether she has enough incentive to choose durian for the sole purpose of experimentation. How hungry she is and how much time she has might be important in whether she prefers the safe option of fish and chips, or the high-risk-high-reward option that is durian. The assumption that interim preferences can be treated as exogenous and context independent does not seem especially convincing.

Importantly, it also restricts our ability to study the connection between taste uncertainty and the observed paradoxes of choice. This link was established in the literature at least since Cox and Grether (1996) observed that the preference reversal paradox is less prevalent in repeated experiments with incentives. Subsequently, Plott (1996) formulated the discovered preference hypothesis. This hypothesis states that the consumer has some real and well defined preferences, that are ex ante unknown to them and only discovered after experience. In experiments, the subjects are usually asked to make choices that are rarely experienced in everyday life, therefore, Plott (1996) suggests that taste uncertainty might be of importance to our understanding of observed paradoxes.

This connection is supported by empirical studies. Not only the observed choices indeed stabilize in repeated experiments (e.g. Kingsley and Brown 2010, Czajkowski et al. 2015), but a large body of evidence suggests that preference discovery can have far reaching consequences for observed behaviour. These include not only the aforementioned preference reversal (e.g. Cox and Grether 1996, Plott 1996, Butler and Loomes 2007), but also the WTP/WTA disparity (e.g. Plott and

Zeiler 2005, Engelmann and Hollard 2010, Humphrey et al. 2017) and the order effects in stated preference studies (e.g. Day et al. 2012, Carlsson et al. 2012). The results obtained by van de Kuilen (2009) even suggest that preference discovery can account for behavioural effects such as probability weighting, as the elicited probability weighting function converges significantly towards linearity when the respondents are asked to make repeated choices.

At the same time, the nature of the connection between preference discovery and paradoxical behaviour remains unclear, meaning that it is not immediately obvious why, for example, taste uncertainty should lead to the observed preference reversals. Indeed, models of both Piermont et al. (2016) and Cooke (2017) do not predict preference reversal or other behavioural paradoxes that I have mentioned. To the best of my knowledge, Loomes et al. (2009) is the only theoretical contribution that connects preference discovery to the observed behavioural paradoxes, as they show that with additional assumption of reference dependence and loss aversion, taste uncertainty has an impact on WTP/WTA disparity. However, even beside the use of reference dependent prospect theory, their model is very parsimonious, as it only considers a finite state space and does not condition taste uncertainty on consumption.

In order to fill this gap, in this article I study how the preferences are formed under taste uncertainty. I believe that although the consumer choice is procedure dependent and which procedure is applied depends on the circumstances, there is a common cognitive basis for these procedures — namely, the consumer's perception of their own taste, which I propose to consider as probabilistic. In other words, I assume that although the consumer does not know their own preferences, they form some probabilistic beliefs regarding their tastes and update those beliefs accordingly with new information that they obtain from consumption. I model those beliefs using a measure theoretic approach, meaning that I assume that the consumer comes equipped with a probability measure on the space of all permissible preference relations. This measure is intended to serve as a common basis with respect to which the various context dependent choice procedures should be defined. The main contribution of this paper is the construction of a family of probabilistic measures that are directly linked to the information regarding the consumer's own preferences, and that are easy to define in an interpretable way.

I do not model consumer choice directly. Instead, based on the constructed measure I define preference relations that capture the main factors that arise from taste uncertainty and then study their properties. These factors are firstly: the expectations of the consumer regarding the ex post preference between the alternatives under consideration; secondly the risk that arises from different levels of certainty of the consumer's beliefs for different alternatives; and finally how much the consumer expects to learn about their own preferences by the consumption of a given alternative. Based on the properties of these preference relations I explain the connection between taste uncertainty and the preference reversal paradox. I also show that taste uncertainty can serve as a cognitive justification for reference or range dependent models, such as Sugden (2003).

Beside the psychological justification, in many practical applications we are interested in finding an alternative that is optimal in some well defined sense, but not necessarily the alternative that the consumer would choose. In such a situation, having a model of how the consumer perceives their own taste should be more beneficial than a model of consumer choice. One clear example of this kind is personalized recommendation; for example, streaming platforms might be more interested in recommending the consumer a movie with the highest probability of being good enough for a pleasant evening and the consumer staying on the platform longer. My results show, that in the case of a taste uncertain consumer, it generally will be a different movie to the one that is optimal with respect to the expected preference of the consumer. It agrees with empirical studies, for example, the results of Shen and Ball (2011) suggest that taste uncertainty is an important factor in the response to personalized recommendations.

The structure of the article is as follows. After a short reviev of the literatur in section 2, I begin with the most basic definitions and the overall setting of the model in section 3. My theory of a taste uncertain consumer is developed in sections 4–5. Out of those two, section 4 is mostly technical and provides the required construction of the probabilistic measure on the space of preferences. The task in section 5 is the definition and the study of the properties of both the expected preferences and risk preferences. The results in this section show that the properties of indirect learning, meaning the dependence of the consumer's beliefs regarding their own preferences on one another, play a crucial role in, among others, transitivity of conditional preferences and their reference dependence. Under the assumption that expected preferences are transitive, I also show that it is possible to represent the measure in a way that is conceptually similar to Gilboa and Schmeidler (1995). Finally, section 6 contains a very short summary of my results, together with a discussion on experimentation in the model and on preference reversal.

2 Literature review

Contemporary literature on taste uncertainty mostly follow Kreps (1979) and use preference for flexibility in order to obtain a subjective state space that represents how the taste uncertain consumer perceives own preferences. Piermont et al. (2016) and Cooke (2017) are two examples of such models. Other existing approaches to taste uncertainty include Loomes et al. (2009) and Jacobson et al. (2014). However, all those models assume that the interim preferences of the consumer are given exogenously. I do not make this assumption and I do not propose any axiomatization of interim preferences. Instead, I provide a language in which we can define any choice procedures that the consumer can reasonably apply.

Conceptually most similar to what I do is the case-based decision theory of Gilboa and Schmeidler (1995). I also consider the consumer that evaluates available alternatives based on the consequences of their past choices, but I apply this idea to the perception of their own taste by the consumer, not to the choice itself. Another important contribution that employ the idea of a consumer that evaluates their options based on an outcome data is Szwagrzak (2022). This article is especially interesting, as it also includes the idea that the consumer is faced with a trade-off between the average utility in the sample and belief certainty, that in this case is a function of the sample size.

My approach is also connected to the literature on the incomplete preferences (e.g. Bewley 2002, Ok et al. 2012, Huang et al. 2014). Indeed the information that is available to the consumer in my model takes the form of an incomplete preference relation. Especially notable from my perspective is the contribution of Huang et al. (2014), as it applies the notion of an evidence distance that is conceptually very close to how I construct the probability measures. The conditional probability measure that is the central object of my model can be understood as a prior beliefs of the consumer over the possible resolution of the incompleteness. In the case of indirect comparisons, every reference point induces a prior probability distribution. As the indirect preferences are inherently reference dependent, my model naturally connects to the literature on multiple prior models (e.g. Gilboa and Schmeidler 1989, Epstein and Schneider 2003, Hanany and Kilbanoff 2007, Gilboa et al. 2010).

The decision not to model the consumer choice directly is motivated mostly by the literature on the construction of the preferences, of which Lichtenstein and Slovic (2006) offers a comprehensive review. However, I do not go this far, as I do not reject the very existence of the preferences. The three preference rankings that I consider are merely a reflection of the fact that under taste uncertainty the alternatives can be evaluated not only by the anticipated preference between them but also by how certain the consumer is of their preference and by the value of information that they expect to learn. This is not a new observation; the trade-off between expected preference, the level of belief certainty and the experimentation is already present in the literature on the multi-armed bandit problem. Important examples include Rothschild (1974), Weitzman (1979), Keller and Rady (1999) and Keller et al. (2019).

Outside of the articles that I have already mentioned, there is still a lot of other contemporary works with important similarities to what I do. It is not feasible to mention every such a contribution, but especially notable are firstly the model of Karni and Viero (2017) that allows for unforeseen consequences of an action. The consumer is anticipating the possibility that they might be ignorant, and manifest this in their choice. Secondly, Wilson (2018) proposes a model of a consumer with preference discovery costs. In this model, the consumer is endowed with a preference relation, but does not optimize perfectly because it demands introspection that is costly. Finally, Wolff and Bauer (2018) propose a model of a belief uncertain consumer that predicts a higher ratio of suboptimal choices when the consumer is presented with an unknown choice.

3 Elementary definitions

Let $\mathcal{B}$ be a space of objects of choice. I assume $\mathcal{B}$ is a compact, connected and metrizable topological space. Generic elements of $\mathcal{B}$ are x, y, z. The interpretation of the metric on $\mathcal{B}$, which I denote by d, is as a measure of similarity, meaning that if $d(x,y) < d(x,z)$ then I interpret x as more similar[1] to y than to z. Abusing notation a little, I also denote by d a product metric on $\mathcal{B} \times \mathcal{B}$ given by $d((x_1,y_1),(x_2,y_2)) = \sqrt{d(x_1,x_2)^2 + d(y_1,y_2)^2}$. Open balls in $\mathcal{B}$ with the centre at x and the radius r are given by $B(x,r)$.

I define Ω to be a set of all permissible preference relations, meaning a set of all binary relations on $\mathcal{B}$ that satisfy axioms 1–3. The generic element of Ω is given by ω. I denote the relation of weak preference with respect to $\omega \in \Omega$ by $x \succeq_\omega y$ and similarly for strict preference and indifference relations I use $\succ_\omega$ and $\sim_\omega$ respectively. I distinguish one special element $\omega^* \in \Omega$ and interpret it as the unknown, real preferences of the consumer.

Axiom 1. *(Rationality) Let $\omega \in \Omega$. Then ω is complete, reflexive and transitive.*

Axiom 2. *(Continuity) Let $\omega \in \Omega$. For each $x \in \mathcal{B}$ sets $\{y \in \mathcal{B} : x \succ_\omega y\}$, $\{y \in \mathcal{B} : y \succ_\omega x\}$ are open.*

Axiom 3. *(Limited Indifference) Let $\omega \in \Omega$. For any $x \in \mathcal{B}$ set $\{y \in \mathcal{B} : x \sim_\omega y\}$ has an empty interior.*

Axioms 1 and 2 are standard axioms of utility theory. As $\mathcal{B}$ is metrizable and compact, it is second countable and as such the theorem of Debreu (1964) states that preferences that satisfy those two axioms can be represented by a continuous utility function. Axiom 3 is the only non-standard axiom that I assume. Its intended interpretation is that for a randomly chosen $x, y \in \mathcal{B}$ and any $\omega \in \Omega$ it is very unlikely that $x \sim_\omega y$.

Let $\mathcal{D} \subset \mathcal{B}$ denote a finite subset of alternatives. I interpret elements of $\mathcal{D}$ as the alternatives that the consumer has already experienced and assume, that the real preference ranking of those elements is known, meaning that for each pair $x, y \in \mathcal{D}$ the consumer knows what is the relation between those two alternatives with respect to ω^*. I define a set $\mathcal{K} = \{xRy : x, y \in \mathcal{D},\ xR_{\omega^*}y\}$, so the set that consists of all these known relations between elements of $\mathcal{D}$. Technically, the elements of $\mathcal{K}$ are formal expressions[2] of form xRy, where $x, y \in \mathcal{B}$ and the relation

[1]Technically d measures dissimilarity of the alternatives, but I call it a measure of similarity nevertheless.

[2]One can also think of $\mathcal{K}$ as a partial preference relation $\mathcal{K} = \omega^*_{|\mathcal{D}}$, meaning a restriction of ω^* to $\mathcal{D}$. In this interpretation, $\Omega(\mathcal{K})$ is a set of all extensions of $\mathcal{K}$ to $\mathcal{B}$. I prefer to treat it as formal expressions in order to have more flexibility in notation.

symbol $R \in \{\succ, \succeq, \sim, \preceq, \prec\}$. Finally, I denote by $\Omega(\mathcal{K}) \subset \Omega$ those preference relations, that agree with ω^* on $\mathcal{D}$, that is $\Omega(\mathcal{K}) = \{\omega \in \Omega : \forall_{xRy \in \mathcal{K}}\ xR_\omega y\}$.

In order to familiarize oneself with all these definitions, consider Example 1 below.

Example 1. Bob invited Alice over for a movie and considers which one to choose. His choice alternatives are different movies, that are uniquely and precisely characterized by n-element vectors of their characteristics normalised to $[0, 1]$, so that $\mathcal{B} = [0, 1]^n$. For each movie Bob infers its characteristics from its description and trailer, and perceives the similarity of those movies based on the euclidean distance between the vectors of their characteristics.

Bob is not a huge movie enthusiast, he has only seen three movies in his life: "Godzilla", "Rambo" and "Titanic" (G, R and T respectively), meaning that $\mathcal{D} = \{G, R, T\}$. Out of those, Godzilla is the movie that he liked the most, Rambo the least, with Titanic in between, so $\mathcal{K} = \{G \succ R, G \succ T, T \succ R\}$ and even though he does not know his real preferences ω^* he knows it must satisfy these relations. He is also aware, that his preferences must be "sensible", where sensible is defined as satisfying axioms 1–3, so ω^* must be an element of $\Omega(\mathcal{K})$.

There are two important things to note from example 1. The first is that ω^* is the only element of the model that is interpreted as unknown to the consumer. I not only assume that the consumer is aware of all the alternative choices, remembers their past choices and ranks those choices without mistakes, but their knowledge goes even deeper than that as they are also aware of axioms 1–3. It means, that they are aware that they are, for example, unlikely to find a movie that is exactly as good as Godzilla, or that if they prefer Godzilla to Rambo, then movies like Godzilla 2 that are sufficiently similar to Godzilla should also be preferred to Rambo. Most importantly, I also assume that the consumer correctly perceives similarities between all alternatives, even those not yet consumed. Consumption only reveals the consumer's preference with respect to the alternative and not anything new regarding the alternative itself, so if the consumer perceives Godzilla 2 to be extremely similar to Godzilla then this perception must remain unchanged after watching Godzilla 2. It is possible that they find their tastes for Godzilla and Godzilla 2 to be starkly different, but this would be the consequence of differences between the movies that they previously knew of but did not know their preferences for, rather than the other way around. The second thing to note is that metric d should be interpreted as the subjective perception of similarity by the consumer, and as such it should usually be treated as unknown in experimental studies. Moreover, this perception of distance can potentially be malleable when faced with framing or other marketing techniques (e.g. Mandel and Johnson 2002, Ariely et al. 2006). Empirical results of Levin and Gaeth (1988) show that the magnitude of framing effect in marketing lessened after the consumption took place, which suggests that the perception of distance is likely to be learned and not ex ante known.

Definitions of ω^* and $\mathcal{D}$ are actually not essential for my model. I state those definitions for the purpose of storytelling, and the definition of $\mathcal{K}$ is the only place where these are employed. In general, I can take $\mathcal{K}$ to be any consistent and transitive set of relations[3] — it does not even have to be complete, for example, $\mathcal{K} = \{x_1 \succ x_2, x_3 \succ x_4\}$ is fine.

4 Construction of measures

The last of the primitives of the model, and the only that I did not introduce in section 3 is a probabilistic measure (denoted by μ) on the set Ω. The intended interpretation of $\mu(A)$ for a measurable $A \subset \Omega$ is a probability of $\omega^* \in A$. I assume the existence of μ in precisely the same way as is typical of preferences, namely, I assume that the consumer comes equipped with μ. However, defining measures on highly abstract sets such as Ω is a non-trivial exercise — and in this case especially so, as I need μ to be connected to the information available to the consumer. For this reason, in this section, I focus on the construction of an appropriate class of measures.

In order to connect information provided by set $\mathcal{K}$ to measure μ I equip Ω

[3] I could be even more general. Using the language of first order logic I can interpret formal expressions like xRy as atomic formulas in some logical signature. I could then allow $\mathcal{K}$ to be any set of well-formed, closed logical formulas in this signature. It is possible to add additional axioms in this way, for example if $\mathcal{B}$ is equipped with some natural ordering, I can add the monotonicity axiom by adding the expression $\forall_{x,y}(x \geq y) \implies (x \succeq y)$ to $\mathcal{K}$. All my results naturally translate to this extended setting. I refrain from this as it would complicate notations and the proofs would need to be longer and less elegant. An interested reader should have no difficulty in extending the model in this way with guidance from any introduction to mathematical logic.

with a topology. Topological constructions on the space of preference relations are nothing new in economics; one well-known example of such a construction is given in Kannai (1970). However, I need a different construction. The following definition 1 is the main building block of this approach.

Definition 1. *Let $x, y \in \mathcal{B}$ and $R \in \{\succ, \succeq, \sim, \prec, \preceq\}$. I define $[xRy] = \{\omega \in \Omega : xR_\omega y\}$ and call it a condition on x, y.*

Note, that μ restricted to conditions, for example $[x \succ y]$ allows for another (equivalent) interpretation, meaning that $\mu([x \succ y])$ is the ex-ante probability that after experiencing both x and y the consumer finds that x is indeed preferred to y. For an arbitrary finite sequence of conditions I denote the intersection and the union of those conditions as respectively $[\bigwedge_{i=1}^n x_i R_i y_i]$ and $[\bigvee_{i=1}^n x_i R_i y_i]$ and call them respectively conjunction and disjunction of conditions. The terms conjunction and disjunction are justified, since

$$\left[\bigwedge_{i=1}^n x_i R_i y_i\right] = \bigcap_{i=1}^n [x_i R_i y_i] = \left\{\omega \in \Omega : \bigwedge_{i=1}^n x_i R_{i\omega} y_i\right\},$$

and similarly

$$\left[\bigvee_{i=1}^n x_i R_i y_i\right] = \bigcup_{i=1}^n [x_i R_i y_i] = \left\{\omega \in \Omega : \bigvee_{i=1}^n x_i R_{i\omega} y_i\right\}.$$

I sometimes write conjunctions of conditions as chains, so for example, I usually write $[x_1 \succeq x_2 \succeq x_3]$ instead of $[x_1 \succeq x_2 \wedge x_2 \succeq x_3]$. For a given conjunction of conditions $[\bigwedge_{i=1}^n x_i R_i y_i]$, I define its length as $l([\bigwedge_{i=1}^n x_i R_i y_i]) = n$ and a set $\mathrm{cp}([\bigwedge_{i=1}^n x_i R_i y_i]) = \{(x_i, y_i) : i \leq n\}$ of the pairs on which the conditions are imposed.

I define topology $\mathcal{T}$ on Ω to be a topology generated by a family of strict conditions, meaning $\{[x \succ y] : x, y \in \mathcal{B}\}$. As a consequence, closed sets are those generated by weak conditions.[4]

[4]This topology is actually very natural. Consider for each $\omega \in \Omega$ the function $f_\omega : \mathcal{B}^2 \to \{-1, 0, 1\}$, such that $f_\omega(x, y) = 1$ iff $x \succ y$, $f_\omega(x, y) = 0$ iff $x \sim y$ and $f_\omega(x, y) = -1$ iff $x \prec y$. Equip the set $\{-1, 0, 1\}$ in the topology, such that open sets are $\emptyset, \{-1\}, \{1\}, \{-1, 1\}, \{-1, 0, 1\}$. With this definition $\omega \in \Omega$ is continuous if and only if f_ω is. Now equip the whole space of continuous functions from $\mathcal{B}^2$ into $\{-1, 0, 1\}$ in the standard product topology and embed Ω into this space using the identification $\omega \to f_\omega$. It is an easy exercise to see that in this way I get the same topology.

At this point I am ready to turn to the construction of μ and the corresponding σ-field. Starting with σ-field, it is clearly a very important precondition for information — meaning conditions and their conjunctions/disjunctions — to be measurable. Proposition 1 gives an obvious condition that is needed for this to happen.

Proposition 1. *Let σ_B denote a Borel σ-field on Ω and σ' be an arbitrary σ-field such that $\forall_{x,y\in\mathcal{B}} : [x \succ y] \in \sigma'$. Then $\sigma_B \subset \sigma'$.*

Proof. Note, that the family of all finite conjunctions of conditions is a base of the topology on Ω. Moreover, as $\mathcal{B}$ is second countable, it is separable and as such it has a countable dense subset. Let $A \subset \mathcal{B}$ be this subset. By axiom 2 each $\omega \in \Omega$ is continuous and as such is uniquely determined by its relations on $A \times A$. Therefore the family of all finite conjunctions of conditions on $A \times A$ also is a base of the topology on Ω. Now it follows that $\sigma(\{[x \succ y] : x, y \in \mathcal{B}\}) = \sigma_B$.

□

Defining a measure on the σ-field of Borel sets is a typical way of ensuring that the measure is in some sense compatible with the topology of the underlying space and proposition 1 states, that it is necessary and sufficient for information to be measurable. From now on I am only interested in μ defined on a Borel σ-field and assume such σ-field to be given.

Now I move on to the construction of μ. Axioms 4–6 state the primitive assumptions that I make about the measure.

Axiom 4. *(Non-degeneracy) Let $U \subset \Omega$ be open and nonempty. Then $\mu(U) > 0$.*

Axiom 5. *(Continuity) For all pairwise non-equal $x, y, z \in \mathcal{B}$, any finite conjunction of conditions α and any $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, z) < \delta \implies |\mu([\alpha \wedge (x \succ y)]) - \mu([\alpha \wedge (z \succ y)])| < \epsilon$.*

Axiom 6. *(Restricted Indifference) Let $x, y \in \mathcal{B}$ be such that $x \neq y$. Then $\mu([x \sim y]) = 0$.*

Axiom 4 states, that for any pair $x, y \in \mathcal{B}$ with $x \neq y$ the consumer perceives that both $x \succ y$ and $y \succ x$ are apriori probable, though generally not with the same probability. Axiom 5 is just a typical continuity axiom. It states that μ restricted to conditions of form $[x \succ y]$ and treated as a function of x, y is continuous. However, there are two important details in this axiom. Firstly, the addition of some arbitrary conjunction of conditions α in the statement. It is necessary because I have not defined the conditional measure yet, and I want axiom 5 to ensure the continuity of all conditional measures as well. Secondly, the restriction of axiom 5 to the pairwise non-equal elements. It is necessary, because by axiom 6 for $x \neq y$ there is $\mu([x \sim y]) = 0$, however it is clear that $\mu([x \sim x]) = 1$. So for the case $x = y$ the measure is inherently discontinuous. In order to avoid the necessity of special treatment of the conditions of form $[x \sim x]$ everywhere, from now on I use the convention that $\mu([x \succ x]) = \frac{1}{2}$ and $\mu([x \sim x]) = 0$. With such convention, axiom 5 holds for any x, y, z.

Axiom 6 is introduced only to allow me to ignore indifference relations as a legitimate possibility, which greatly simplifies proofs and notation. This axiom is clearly connected to axiom 3, however it is not implied by it. Axiom 3 restricts the space Ω (its main function is to assure that topology on Ω is nice), whereas axiom 6 restricts measures on Ω.

Definition 2. *Let μ and some measurable $A \subset \Omega$ be given and denote $\bar{K} = \mathcal{K} \setminus \{x \sim y : x \sim y \in \mathcal{K}\}$. I define*

$$\mu_{\mathcal{K}}(A) = \begin{cases} 0, & \text{if } A \cap \Omega(\mathcal{D}) = \emptyset, \\ \frac{\mu(A \cap \Omega(\bar{K}))}{\Omega(\bar{K})}, & \text{otherwise.} \end{cases}$$

In order to understand definition 2 it suffices to note that as long as there is no indifference relations in $\mathcal{K}$, definition 2 is a standard definition of conditional probability. This is, because due to axiom 4, in this case I have $\mu(\Omega(\mathcal{K})) > 0$. However, in case there are some indifference relations in $\mathcal{K}$, I have $\mu(\Omega(\mathcal{K})) = 0$ and I cannot define a probability conditional on a measure zero set. Definition 2 therefore instructs us to simply assign values of conditional probability as if those indifferences were not there — those relations are simply ignored, but the support of $\mu_{\mathcal{K}}$ is still restricted to $\Omega(\mathcal{K})$, not $\Omega(\bar{K})$.

One important implication of this definition is that it explicitly assumes that the conditional measure is path independent, meaning that the only thing that matters are the known relations provided by $\mathcal{K}$, and these are independent from the order in which the alternatives were explored. This is contrary to the assertions in the literature on preference construction (see Lichtenstein and Slovic 2006 for a comprehensive review) and is a consequence of my assumption that ω^* exists and is revealed by consumption.

From this point onwards, the main objective of this section is to obtain theorem 1 that allows us to easily define measures on Ω. The statement of this theorem is preceded by supporting facts and definitions.

Definition 3. *Fix $U \subset \Omega$ and let $K = \{\alpha_i : i \in I\}$ for arbitrary $I \subset \mathbb{N}$ and conjunctions of conditions α_i. K is a representation of U if $U = \bigcup_{i \in I} \alpha_i$.*

Definition 4. *The representation $K = \{\alpha_i : i \in I\}$ is disjoint if $i_1 \neq i_2$ implies that $[\alpha_{i_1}] \cap [\alpha_{i_2}] = \emptyset$.*

Definition 5. *Let K_1, K_2 be two representations of some set $U \subset \Omega$. K_1 is subordinate to K_2 if for all $\alpha \in K_1$ there is $\alpha' \in K_2$ such that $[\alpha] \subset [\alpha']$.*

Definitions 3–5 introduce the concept of representation and its two properties. I mainly work with conditions and their conjunctions/disjunctions and representations allow me to study more sets using the same techniques. It is clear that the representations are not unique, and that not every set has a representation (remember, that I only allow for conjunctions of finitely many conditions) — but this is enough.

Lemma 1. *Let U have a finite representation. Then U has a finite disjoint representation.*

Proof. I proceed by induction on m, which is the number of conjunctions of conditions in the representation of U. Let $m = 1$. Then $U = [\bigwedge_{i=1}^{n_1} x_{i1} R_{i1} y_{i1}]$ and therefore the thesis is trivially satisfied. I just need to prove the implication that if the thesis is satisfied for some m, then it is satisfied for $m + 1$.

Assume, that for any $U = [\bigvee_{j=1}^{m} \bigwedge_{i=1}^{n_j} x_{ij} R_{ij} y_{ij}]$ I have a disjoint representation as $U = [\bigvee_{j=1}^{m'} \bigwedge_{i=1}^{n'_j} x'_{ij} R'_{ij} y'_{ij}]$. Now assume $U = [\bigvee_{j=1}^{m+1} \bigwedge_{i=1}^{n_j} x_{ij} R_{ij} y_{ij}]$. By assumption, I have that $U = [\bigvee_{j=1}^{m'} \bigwedge_{i=1}^{n'_j} x'_{ij} R'_{ij} y'_{ij} \vee \bigwedge_{i=1}^{n_{m+1}} x_{im+1} R_{im+1} y_{im+1}]$. Denote $\gamma = \bigwedge_{i=1}^{n_{m+1}} x_{im+1} R_{im+1} y_{im+1} \wedge \bigwedge_{j=1}^{m'} \neg\alpha_j$, where $\neg\alpha_j = \bigvee_{i=1}^{n'_j} \neg x'_{ij} R'_{ij} y'_{ij}$ and $\neg x R y$ is, for R respectively $\succ, \succeq, \sim$, given by $y \succeq x$, $y \succ x$ and $(y \succ x \vee x \succ y)$, therefore γ is of the required form.

As every logical formula can be rewritten in disjunctive normal form, there exist some $(\alpha_k)_{k=1}^{m_k}$ such that $\alpha_k = \bigwedge_{i=1}^{n'_k} x'_{ik} R'_{ik} y'_{ik}$ and $\gamma = \bigvee_{k=1}^{m_k} \alpha_k$. To finish the proof, it suffices to show that each α_k is disjoint with each α_j, which follows trivially, as by construction it is clear that for each k, j there exists some i and a logical formula ϕ such that I can represent $\alpha_k = \phi \wedge \neg x'_{ij} R'_{ij} y'_{ij}$.

□

Definition 6. *Let $\alpha = \bigwedge_{i=1}^{n} x_i R_i y_i$ be a logical formula and fix $x, y \in \mathcal{B}$. I call a triple of logical formulas $\alpha \wedge (x \succ y), \alpha \wedge (y \succ x), \alpha \wedge (y \sim x)$ the partition of α by x, y. Moreover, let a finite disjoint representation K of some $U \subset \Omega$ be given, together with a finite set $A \subset \mathcal{B} \times \mathcal{B}$. $\tilde{K}$ is a full partition of K with respect to A if $\tilde{K}$ represents U and every $\alpha' \in \tilde{K}$ is of form $\alpha' = \alpha \wedge \left(\bigwedge_{(x,y) \in A} x R_{xy} y\right)$ where $\alpha \in K$ and $R_{xy} \in \{\succ, \prec, \sim\}$.*

Definition 7. *Let logical formulas $\gamma_1, \gamma_2, \gamma_3$ of form $\alpha \wedge (x \succ y), \alpha \wedge (y \succ x), \alpha \wedge (x \sim y)$ be given. I call logical formula α the merger of $\gamma_1, \gamma_2, \gamma_3$. Moreover $K_1 = \{\alpha\}$ is a full merger of $K_2 = \{\alpha_1, \ldots, \alpha_n\}$ for $n > 1$ if K_2 is a disjoint representation of $U = [\alpha]$.*

Definition 8. *Let conjunction of conditions $\alpha = \bigwedge_{i=1}^{n} x_i R_i y_i$ and a finite set $A \subset \mathcal{B} \times \mathcal{B}$ be given. I define restriction of α to A as rest$(\alpha, A) = \bigwedge_{(x_i, y_i) \in A} x_i R_i y_i$.*

Definitions 6-8 state the main operations that I apply to work with representations of sets. Of special importance are definitions 6 and 7. As shown by lemma 2 and 3 those operations allow us to connect different representations of the same set.

Lemma 2. *Let $U \subset \Omega$ and fix two disjoint representations $K_0 = \{\alpha_1, \ldots, \alpha_n\}$ and $K = \{\alpha'_j : j \in \mathbb{N}_+\}$ of U such that K is subordinate to K_0. There exists a sequence $(K_l)_{l \in \mathbb{N}}$ of representations such that $\bigcap_{k=0}^{\infty} \bigcup_{l=k}^{\infty} K_l = K$ and that K_{l+1} is obtained from K_l using only partitions and mergers (full or otherwise).*

Proof. Fix U, K_0 and K as in the statement of the lemma, and assume all elements of K_0 are enumerated, meaning that $K_0 = \{\alpha_1, \ldots, \alpha_n\}$. I prove this lemma constructively, by providing a procedure to obtain a sequence of representations that satisfy the conditions of the lemma. For each $l \in \mathbb{N}$ I do the following steps.

Fix $\alpha_i \in K_l$, with $i = 1$ in case $l = 1$. I also fix j, starting with $j = 1$ for $l = 1$. Define $K_{\alpha_i} = \{\alpha' \in K : [\alpha'] \subset [\alpha_i]\}$ and $K_{\alpha_i}(n) = \{\alpha' \in K_{\alpha_i} : l(\alpha') = n\}$. Fix n to be the smallest number such that $K_{\alpha_i}(n) \neq \emptyset$ and denote

$$D_{\alpha_i}(n) = \bigcup_{\alpha' \in K_{\alpha_i}(n)} \mathrm{cp}(\alpha') \setminus \mathrm{cp}(\alpha_i),$$

so that $D_{\alpha_i}(n)$ is a set of points on which additional conditions in $K_{\alpha_i}(n)$ are imposed. Note, that $D_{\alpha_i}(n)$ is finite. Define $K^{FP}_{\alpha_i}$ to be the full partition of α with respect to $D_{\alpha_i}(n)$ and define $K^{rest}_{\alpha_i} = \{\text{rest}(\alpha', D_{\alpha_i}(n) \cup cp(\alpha_i)) : \alpha' \in K_{\alpha_i}\}$.

As both $K^{FP}_{\alpha_i}, K^{rest}_{\alpha_i}$ are finite and disjoint and $K^{FP}_{\alpha_i}$ is subordinate to $K^{rest}_{\alpha_i}$, I can obtain each element $\alpha' \in K^{rest}_{\alpha_i}$ by a full merger of all elements of $K^{FP}_{\alpha_i}$ that satisfy $[\alpha''] \subset [\alpha]$. As a result, I can obtain $K^{rest}_{\alpha_i}$ from α using only partitions and mergers.

Finally, define $K_{l+1} = K_l \cup K^{rest}_{\alpha_i} \setminus \{\alpha_i\}$. I do not change the enumeration of elements in K_{l+1}. As such, all elements of $K_{l+1} \cap K^{rest}_{\alpha_i}$ are not enumerated (for now), meaning that there is no i' such that $\alpha_{i'} \in K_{l+1} \cap K^{rest}_{\alpha_i}$. Note that by construction $K_{\alpha_i}(n) \subset K_{l+1}$, K_{l+1} is disjoint and K is subordinate to K_{l+1}.

Now, if $\alpha_{i+1} \in K_{l+1}$, increase i, l by one and perform the same operations as I did up to this point. In the other case, increase l and j by one, set $i = 1$ and enumerate all elements of K_l.

Note, that clearly if $\alpha' \in K$ and $\alpha' \in K_l$ for any l then also $\alpha' \in K_{l+1}$. Therefore in order to finish the proof, I just need show that every element $\alpha' \in K$ is obtained as an element of K_l for some l. Let $[\alpha'] \subset [\alpha]$ for some $\alpha \in K_0$ and fix $n_{\alpha'} = |\{n \leq l(\alpha') : K_\alpha(n) \neq \emptyset\}|$.

I claim, that I obtain α' as an element of K_l for some l such that $j = n_{\alpha'}$. Indeed, if $n_{\alpha'} = 1$, I have already shown it. Consider $n_{\alpha'} > 1$. There is i such that $[\alpha'] \subset [\alpha_i]$ for $j = 1$ and there is $\alpha'' \in K^{rest}_{\alpha_i}$ such that $[\alpha'] \subset [\alpha'']$. As $K_{\alpha''} \subset K_{\alpha_i} \setminus K_{\alpha_i}(n)$ where n is the smallest number such that $K_{\alpha_i}(n) \neq \emptyset$ I get that $|\{n \leq l(\alpha') : K_{\alpha''}(n) \neq \emptyset\}| < n_{\alpha'}$, proving the claim. Sine for each j I perform a finite number of partitions and mergers, the proof is complete.

□

Lemma 3. *Fix two finite disjoint representations $K_1 = \{\alpha_1, \ldots, \alpha_{m_1}\}$ and $K_2 = \{\alpha'_1, \ldots, \alpha'_{m_2}\}$ of some set $A \subset \Omega$. Assume that set function*

$$\mu_0 : \{A \subset \Omega : \exists_{x_i, y_i, n}\ A = [\bigwedge_{i=1}^{n} x_i \succ y_i]\} \to [0, 1]$$

that for any conjunction of conditions α satisfies both $\mu_0([\alpha \wedge (x \sim y)]) = 0$ and $\mu_0([\alpha \wedge (x \succ y)]) + \mu_0([\alpha \wedge (y \succ x)]) = \mu_0([\alpha])$ is given. Then $\sum_{j=1}^{m_1} \mu_0([\alpha_j]) = \sum_{j=1}^{m_2} \mu_0([\alpha'_j])$.

Proof. By the condition that $\mu_0([\alpha \wedge x \succ y]) + \mu_0([\alpha \wedge y \succ x]) = \mu_0([\alpha])$ the values of μ_0 are assigned in such a way that replacing any α_{j_0} or α'_{j_0} by its arbitrary partition, for example replacing α_{j_0} by $\alpha^1_{j_0}, \alpha^2_{j_0}$ gives $\sum_{j=1}^{m_1} \mu_0([\alpha_j]) = \mu_0([\alpha^1_{j_0}]) + \mu_0([\alpha^2_{j_0}]) + \sum_{j \neq j_0}^{m_1} \mu_0([\alpha_j])$. Therefore it suffices to show, that there exists a finite sequence of partitions from both K_1 and K_2 to some $K = \{\alpha_1^{l_1}, \ldots, \alpha_{k_1}^{l_1}\}$, meaning I can obtain the same finite subset of formulas as a result of the recursive partitioning of K_1 and K_2. It suffices to define

$$D = \bigcup_{j=1}^{m_1} \mathrm{cp}(\alpha_j) \cup \bigcup_{j'=1}^{m_2} \mathrm{cp}(\alpha'_{j'}),$$

and fix K to be a representation obtained by partitioning of all formulas in K_1 on all elements of D. Obviously, partitioning all elements of K_2 on all elements of D I also obtain K.

□

I am now ready to state and prove the main result of this section.

Theorem 1. *Assume, that for all $n \in \mathbb{N}_+$ and all sequences $(x_i, y_i)_{i=1}^n \in \mathcal{B} \times \mathcal{B}$ the values of set function $\mu_0([\bigwedge_{i=1}^n x_i \succ y_i]) > 0$ are given and satisfy*

$$\mu_0([\bigwedge_{i=1}^{n-1} x_i \succ y_i \wedge x_i \succ y_i]) + \mu_0([\bigwedge_{i=1}^{n-1} x_i \succ y_i \wedge y_i \succ x_i]) = \mu_0([\bigwedge_{i=1}^{n-1} x_i \succ y_i),$$

and

$$\mu_0([x \succ y]) + \mu_0([y \succ x]) = 1.$$

There then exists a unique probabilistic measure μ defined on the whole Borel σ-field of Ω such that for all conjunctions of conditions $\mu_0([\bigwedge_{i=1}^n x_i \succ y_i]) = \mu([\bigwedge_{i=1}^n x_i \succ y_i])$.

Proof. Define a family of sets

$$\mathcal{A} = \left\{ [\bigvee_{j=1}^{m} \bigwedge_{i=1}^{n_j} x_{ij} R_{ij} y_{ij}] : x_{ij}, y_{ij} \in \mathcal{B}, R_{ij} \in \{\succ, \succeq, \sim\} \right\} \cup \emptyset.$$

I show, that $\mathcal{A}$ is an algebra of sets. It contains an empty set, and is obviously closed under binary unions. Moreover, it is closed under complementation, as

$$[\bigvee_{j=1}^{m} \bigwedge_{i=1}^{n_j} x_{ij} R_{ij} y_{ij}] \backslash [\bigvee_{j=1}^{m'} \bigwedge_{i=1}^{n'_j} x'_{ij} R'_{ij} y'_{ij}] = \left[\bigvee_{j=1}^{m} \left(\bigwedge_{i=1}^{n_j} x_{ij} R_{ij} y_{ij} \wedge \bigwedge_{j=1}^{m'} \bigvee_{i=1}^{n'_j} (\neg x'_{ij} R'_{ij} y'_{ij}) \right) \right] = A,$$

where $\neg x'_{ij} \succ y'_{ij} = y'_{ij} \succeq x'_{ij}$, $\neg x'_{ij} \succeq y'_{ij} = y'_{ij} \succ x'_{ij}$ and $\neg x'_{ij} \sim y'_{ij} = y'_{ij} \succ x'_{ij} \vee x'_{ij} \succ y'_{ij}$. Since every logical formula can be stated in disjunctive normal form, $A \in \mathcal{A}$ and therefor $\mathcal{A}$ is an algebra of sets.

I first extend μ_0 to the whole $\mathcal{A}$ as follows: define $\mu_0([\bigwedge_{i=1}^n x_i R_i y_i]) = 0$ if any $R_i = \sim$ and $\mu_0([\bigwedge_{i=1}^n x_i R_i y_i]) = \mu_0([\bigwedge_{i=1}^n x_i \succ y_i])$ otherwise. Moreover from proposition 1 I get that each $A \in \mathcal{A}$ can be represented by some $[\bigvee_{j=1}^m \bigwedge_{i=1}^{n_j} x_{ij} \succ y_{ij}]$ that is disjoint. Therefore I define $\mu_0(A) = \sum_{j=1}^m \mu_0([\bigwedge_{i=1}^{n_j} x_{ij} \succ y_{ij}])$. Note that this is well defined due to lemma 3, which I can apply due to the condition in the statement of the theorem. This is therefore a unique extension of μ_0 to $\mathcal{A}$ such that the extended μ_0 is finitely additive.

I need to show, that μ_0 is a pre-measure on $\mathcal{A}$. Fix some $A \in \mathcal{A}$ and let $(A_j)_{j=1}^\infty$, $A_j \in \mathcal{A}$ be disjoint and such that $\bigcup_{j=1}^\infty A_j = A$. Moreover, denote by K the representation of A corresponding to A_j's, so that $K = \{\alpha_j : j \in \mathbb{N}_+\}$. I need to show that $\mu_0(A) = \sum_{j=1}^\infty \mu_0(A_j)$.

Let $\tilde{K_0}$ be an arbitrary disjoint representation of A. By lemma 1 some disjoint representation exists. Define $D = \bigcup_{\alpha \in \tilde{K_0}} \mathrm{cp}(\alpha)$ and take $K_0 = \{\mathrm{rest}(\alpha) : \alpha \in K\}$. Clearly, K_0 is finite and K is subordinate to K_0. Therefore from lemma 2 I have that there exists a sequence of representations $(K^l)_{l \in \mathbb{N}}$ such that K^{l+1} is obtained from K_l using mergers and partitions only, and that $\bigcap_{k=0}^\infty \bigcup_{l=k}^\infty K^l = K$, so the limit of this sequence of recursive partitions and mergers is K. Note, that by finite additivity of μ_0 mergers and partitions have no impact, meaning that for every $l \in \mathbb{N}$

$$\sum_{\alpha \in K^l} \mu_0([\alpha]) = \mu_0(A).$$

Now consider two sequences $m_l = \sum_{\alpha \in K^l} \mu_0([\alpha])$ and $m^l = \mu_0([\bigcap_{k=0}^l \bigcup_{k'=k}^l K^{k'}])$. It is clear that both m_l and m^l are constant and equal to $\mu_0(A)$. Therefore $\lim_{l \to \infty} m_l = \mu_0(A) = \lim_{l \to \infty} m^l$. It now suffices to note, that $\lim_{l \to \infty} m_l = \sum_{\alpha \in K} \mu_0([\alpha])$ and $\lim_{l \to \infty} m^l = \lim_{l \to \infty} \mu_0([\bigcap_{k=0}^l \bigcup_{k'=k}^l K^{k'}]) = \mu_0(\bigcup_{\alpha \in K} [\alpha]) = \mu_0(A)$. Therefore μ_0 is a pre-measure on $\mathcal{A}$.

As $\mathcal{A}$ is an algebra of sets and μ_0 is a finite pre-measure that is uniquely extended to $\mathcal{A}$ from given values, then by Caratheodory's extension theorem it follows, that there exists a unique σ-finite measure μ that extends μ_0 to the whole σ-field generated by $\mathcal{A}$. As $\mathcal{A}$ contains the generating set of the topology on Ω, the σ-field generated by it must contain all open sets, and as a consequence all Borel sets. To finish the proof it suffices to show that μ is probabilistic, but this follows trivially from the condition that $\mu_0([x \succ y]) + \mu_0([y \succ x]) = 1$.

□

Theorem 1 shows that I can define probabilistic measures on the Borel subsets of Ω by specifying their values on a much smaller subset, that consists only of conjunctions of strict conditions. This is a way to specify the measure that is easy to understand and interpret, since conjunctions of conditions represent the joint knowledge of the consumer. In other words, I define a measure by specifying how prior beliefs about the consumer's own tastes are dependent on one another. As a consequence, theorem 1 indirectly tells us that there are no bounds on indirect learning in the model, since it does not impose any additional restrictions on the dependence structure. The only restriction is that the indirect learning is continuous, as stated in axiom 5.

Another consequence of theorem 1, presented in corollary 1 is that I can define conditional measures $\mu_{\mathcal{K}}$ independently of one another.

Corollary 1. *Let for all $x, y \in \mathcal{B}$ and for all sets of conditions $\mathcal{K}$ values of $\mu^0_{\mathcal{K}}([x \succ y]) > 0$ be given and satisfy*

$$\mu^0_{\mathcal{K}}([x \succ y]) + \mu^0_{\mathcal{K}}([y \succ x]) = 1, \quad \mu^0_{\emptyset}([x \succ y]) + \mu^0_{\emptyset}([y \succ x]) = 1.$$

There exists a unique probabilistic measure μ defined on the whole Borel σ-field, such that for all x, y there is $\mu^0_{\emptyset}([x \succ y]) = \mu([x \succ y])$ and $\mu^0_{\mathcal{K}}([x \succ y]) = \mu_{\mathcal{K}}([x \succ y])$.

Proof. Note that $\mu([\bigwedge_{\alpha \in \mathcal{K}} \alpha \wedge x \succ y]) = \mu_{\mathcal{K}}([x \succ y])\mu(\Omega(\mathcal{K}))$. Now for it to follow from theorem 1 I just need to show that I am able to calculate $\mu(\Omega(\mathcal{K}))$ using values of $\mu_{\mathcal{K}}([x \succ y])$ only. It suffices to do this inductively. Let $\mathcal{K} = \{x_1 \succ x_2\}$. Then $\mu(\Omega(\mathcal{K})) = \mu_{\emptyset}([x_1 \succ x_2])$. Now assume I am given $\mu(\Omega(\mathcal{K}))$ for $\mathcal{K} = \{x_1 \succ \cdots \succ x_n\}$ and let $\mathcal{K}' = \{x_1 \succ \cdots \succ x_{n+1} \succ \cdots \succ x_n\}$. From definition 2 I have $\mu(\Omega(\mathcal{K}')) = \mu(\Omega(\mathcal{K}))\mu_{\mathcal{K}}([x_j \succ x_{n+1} \wedge x_{n+1} \succ x_{j+1}])$, where $\mu_{\mathcal{K}}([x_j \succ x_{n+1} \wedge x_{n+1} \succ x_{j+1}]) = 1 - \mu_{\mathcal{K}}([x_{n+1} \succ x_j \vee x_{j+1} \succ x_{n+1}])$. Given that $x_j \succ x_{j+1} \in \mathcal{K}$, the sets $[x_{n+1} \succ x_j], [x_{j+1} \succ x_{n+1}]$ are disjoint in $\Omega(\mathcal{K})$. Therefore $\mu_{\mathcal{K}}([x_j \succ x_{n+1} \wedge x_{n+1} \succ x_{j+1}]) = 1 - \mu_{\mathcal{K}}([x_{n+1} \succ x_j]) - \mu_{\mathcal{K}}([x_{j+1} \succ x_{n+1}])$.

□

Corollary 1 allows me to safely consider $\mu_{\mathcal{K}}$ for different $\mathcal{K}$ independently, knowing that I can combine those into a single measure on Ω. This result is possible only because axioms 4–6 impose very weak conditions on μ. Certainly not all assignments of $\mu_{\mathcal{K}}$ to $\mathcal{K}$ are reasonable. In the current study it is a positive result, as it shows the generality of the proposed language. However, any further work on this topic that involves dynamic learning of preferences should consider imposing additional restrictions.

5 Conditional Preferences

Conditional preferences is the name that I have chosen for any preference relation obtained from the conditional measure $\mu_{\mathcal{K}}$. Such a relation represents interim preferences of the consumer, conditionally on the set of knowledge $\mathcal{K}$. I agree with Tversky et al. (1988) that the consumer choice is procedure and context dependent, and as such I do not propose any universal choice function or preference relation. Instead, I focus on the identification and properties of the two dimensions of choice that are most clearly distinct in the case of taste uncertainty, namely the consumer's expectations of their own ex post preference, and the perception of risk, by which I mean how certain the consumer is of their taste for a given alternative.

The expected preferences are captured in a very natural way by direct comparisons, as stated in definition 9.

Definition 9. *Let $\mu_{\mathcal{K}}$ be given. The relation $\succeq_{\mathcal{K}}$ defined by $x \succeq_{\mathcal{K}} y \iff \mu_{\mathcal{K}}([x \succeq y]) \geq \frac{1}{2}$ is the expected preference relation.*

For a given μ and $\mathcal{K}$ I also use the symbol $\omega_{\mathcal{K}}$ to denote the expected preference relation, and for a given preference relation ω I say that a measure $\mu_{\mathcal{K}}$ represents ω if $x \succeq_{\omega} y \iff \mu_{\mathcal{K}}([x \succeq y]) \geq \frac{1}{2}$. Definition 9 states that $x \succeq_{\mathcal{K}} y$ if and only if the consumer believes that ex post it most probably will be the case that $x \succeq_{\omega^*} y$. Equivalently, it can be understood as saying that most (measured by μ) of the preferences that agree with $\mathcal{K}$ satisfy $x \succeq y$. For the time being, I do not know whether $\omega_{\mathcal{K}}$ is actually a preference relation as it is not immediately obvious that it is transitive.

It is less straightforward how to capture the risk perception. I propose to do it using indirect comparisons, as stated by definition 10.

Definition 10. *Let $z \in \mathcal{B}$. The indirect preference relation $\succeq_z$ with respect to reference point z is defined as $x \succeq_z y \iff \mu_{\mathcal{K}}([x \succeq z]) \geq \mu_{\mathcal{K}}([y \succeq z])$.*

I also denote the relation $\succeq_z$ as ω_z. Note, that by definition $u_z(x) = \mu_{\mathcal{K}}([x \succeq z])$ is a continuous utility function representing ω_z, so it is a transitive and continuous preference relation. The reason why this relation captures the risk perception is best summarised by the following example 2.

Example 2. Bob has narrowed his search for the movie down to two movies: "Godzilla 2" (denoted G_2) and "Gone with the wind" (W). He perceives "Gone with the wind" as most likely the better movie for a date with Alice, meaning that $\mu_{\mathcal{K}}([W \succ G_2]) > \frac{1}{2}$. However he has never seen any movie that is similar to "Gone with the wind" and is worried that it can turn out to be boring. Given the set $\mathcal{K}$ as defined in the example 1, he believes this movie to satisfy $\mu_{\mathcal{K}}([W \succ G]) = \frac{2}{3}$ and $\mu_{\mathcal{K}}([R \succ W]) = \frac{1}{3}$, whereas he is pretty certain that "Godzilla 2" will be similar to "Godzilla", so that $\mu_{\mathcal{K}}([G_2 \succ G]) = \frac{1}{2}$ and $\mu_{\mathcal{K}}([G \succ G_2 \succ T]) = \frac{1}{2}$.

Bob is risk averse and thinks the date will be a success only if the movie is at least as good as "Titanic". As the probability $\mu_{\mathcal{K}}([W \succ T]) = \frac{2}{3}$ and $\mu_{\mathcal{K}}([G2 \succ T]) = 1$, he prefers to choose "Godzilla 2" as it is the least risky option.

Note, that $f_x(z) = \mu_{\mathcal{K}}([x \succ z])$ can be treated as a cumulative distribution function for a given $x \in \mathcal{B}$, meaning that $f_x(z)$ returns a probability of x being at least as good as z. However, comparison of the whole distribution is conceptually difficult, outside of the case of stochastic dominance. Indirect comparison of x, y using z as a reference point compares f_x, f_y evaluated at the point z, and therefore allows to at least partially capture the risk associated with some alternative x. In order to have a better overview of the risk associated with those alternatives, multiple reference points should be used. However, example 2 shows that even the use of one, carefully chosen reference point is able to capture the risk that the consumer is actually interested in.

The indirect preference relation is also of interest because of how it relates to personalized recommendations. In most applications, the objective of personalized recommendations is to maximize the probability of the consumer being interested enough to engage with the recommendation, for example, by clicking on the ad. If reference point z is fixed to be some alternative that we think of as "just interesting enough for the consumer to engage", then maximizing the probability of engagement is equivalent to the maximization of ω_z.

Note, that for indirect comparisons to be interpretable in this way, it is necessary that for z_1, z_2 such that $z_1 \sim_{\mathcal{K}} z_2$, preference rankings $\omega_{z_1}, \omega_{z_2}$ coincide[5]. As a consequense, two natural questions arise beside the transitivity of $\omega_{\mathcal{K}}$, namely when indirect comparison lead to the same choice as direct comparison, and when the reference point z matters. Definition 11 is a crucial building block of my approach to these questions.

Definition 11. *$\mu_{\mathcal{K}}$ is coherent if* $(\mu_{\mathcal{K}}([x \succeq y]) \geq \frac{1}{2}) \implies (\forall_{z \in \mathcal{B}} : \mu_{\mathcal{K}}([x \succeq z \succeq y]) \geq \mu_{\mathcal{K}}([y \succeq z \succeq x]))$.

Coherence demands, that if the consumer believes x to be better than y, then this relation cannot reverse when some unrelated z is added in between. This is a restriction on the indirect learning on the consumer. Clearly, coherence also connects direct comparison between x, y with indirect comparison that use z as a reference point. This connection is made precise by the proposition 2.

Proposition 2. *Let $z \in \mathcal{B}$ and $\mathcal{K}$ be fixed. Then*

1. *$\forall_{z' \in \mathcal{B}} : \omega_z = \omega_{z'}$ if and only if $\mu_{\mathcal{K}}$ is coherent.*

2. *$\forall_{z \in \mathcal{B}} : \omega_z = \omega_{\mathcal{K}}$ if and only if $\mu_{\mathcal{K}}$ is coherent.*

Proof. Assume $\mu_{\mathcal{K}}$ is coherent and without loss of generality fix $x, y \in \mathcal{B}$ such that $\mu_{\mathcal{K}}([x \succeq y]) \geq \frac{1}{2}$. First note, that coherence is equivalent to the condition, that $\mu_{\mathcal{K}}([x \succ y]) \geq \frac{1}{2} \implies \forall_{z \in \mathcal{B}} \mu_{\mathcal{K}}([x \succ z]) \geq \mu_{\mathcal{K}}([y \succ z])$. Indeed

$$\mu_{\mathcal{K}}([x \succ z]) \geq \mu_{\mathcal{K}}([y \succ z]) \iff \mu_{\mathcal{K}}([y \succ x \succ z]) + \mu_{\mathcal{K}}([x \succ y \succ z]) + \mu_{\mathcal{K}}([x \succ z \succ y]) \geq$$

$$\geq \mu_{\mathcal{K}}([x \succ y \succ z]) + \mu_{\mathcal{K}}([y \succ x \succ z]) + \mu_{\mathcal{K}}([y \succ z \succ x]) \iff$$

$$\iff \mu_{\mathcal{K}}([x \succ z \succ y]) \geq \mu_{\mathcal{K}}([y \succ z \succ x]).$$

Therefore coherence is equivalent to the fact, that for any $z_1, z_2 \in \mathcal{B}$ I have that $u_{z_1}(x) \geq u_{z_1}(y)$ and $u_{z_2}(x) \geq u_{z_2}(y)$, and therefore $\omega_{z_1} = \omega_{z_2}$. Now assume that $\forall_{z_1, z_2 \in \mathcal{B}} : \omega_{z_1} = \omega_{z_2}$. I will show that $\mu_{\mathcal{K}}$ is coherent. Again fix $x, y \in \mathcal{B}$ such that $\mu_{\mathcal{K}}([x \succeq y]) \geq \frac{1}{2}$. Note that for any $z \in \mathcal{B}$, u_z and u_y represent the same preferences. Since $u_y(x) \geq u_y(y)$, therefore $u_z(x) \geq u_z(y)$, and therefore coherence.

Now assume that $\forall_{z \in \mathcal{B}} : \omega_z = \omega_{\mathcal{K}}$. Therefore especially for any $z_1, z_2 \in \mathcal{B}$ I have $\omega_{z_1} = \omega_{z_2}$, therefore following point 2 $\mu_{\mathcal{K}}$ is coherent. Now assume $\mu_{\mathcal{K}}$ is coherent,

[5] It seems an especially natural reqiurement in the case of $z_1 \sim z_2 \in \mathcal{K}$

so from point 2 for all $z_1, z_2 \in \mathcal{B}$ I have that $\omega_{z_1} = \omega_{z_2}$. Therefore especially for any z I have $\omega_y = \omega_z$ and $\mu_{\mathcal{K}}([x \succeq y]) \geq \frac{1}{2} \implies u_y(x) \geq u_y(y)$, and therefore $\forall_{z \in \mathcal{B}} : \omega_z = \omega_{\mathcal{K}}$.

□

Proposition 2 shows, that coherence is an equivalent to the fact that that indirect comparisons are both equivalent to direct comparisons independent of the reference point. Therefore for coherent $\mu_{\mathcal{K}}$, rankings with respect to risk and expected preference coincide, meaning that it holds, that if I believe that x is better than y then I should believe x has a better chance of being preferred to some z then y does. In such a case, the consumer can be reasonably expected to behave as if under perfect information and simply maximize their expected preference. As shown by the proposition 3 coherence is a very strong property and not easy to satisfy.

Proposition 3. *Let $x_1 \succ x_2, x_2 \succ x_3 \in \mathcal{K}$. Then $\mu_{\mathcal{K}}$ is not coherent.*

Proof. Fix $\mathcal{K}$ as in the statement of the theorem. As $\mu_{\mathcal{K}}([x_1 \succeq x_3]) = 1$ and $\mu_{\mathcal{D}}([x_2 \succeq x_3]) = 1$ from continuity for any disjoint open ball $B(x_1, r_1), B(x_2, r_2) \subset \mathcal{B}$ and for any $z_2 \in B(x_2, r_2)$ there exists $z_1 \in B(x_1, r_1)$ such that $\mu_{\mathcal{D}}([z_2 \succeq x_3]) > \mu_{\mathcal{D}}([z_1 \succeq x_3])$. Therefore from coherence $\mu_{\mathcal{D}}([z_2 \succeq z_1]) \geq \frac{1}{2}$. However $\mu_{\mathcal{D}}([x_1 \succeq x_2]) = 1$ and therefore from continuity there exist disjoint open balls $B(x_1, r_1), B(x_2, r_2) \subset \mathcal{B}$ so that for all $z_1 \in B(x_1, r_1), z_2 \in B(x_2, r_2)$ I have $\mu_{\mathcal{D}}([z_1 \succ z_2]) > \frac{1}{2}$, which is a contradiction.

□

Proposition 3 shows that coherence is not satisfied whenever the consumer knows their real preference between at least two pairs of alternatives. It is so because continuity in axiom 5 implies that indirect learning is not uniform. The consumer learns more about the alternatives that are similar, and the resulting variation in the level of knowledge makes some alternatives more risky.

By proposition 2 coherence is sufficient for the transitivity of $\omega_{\mathcal{K}}$, but it is not a necessary condition. The sufficient and necessary condition is given by the definition 12.

Definition 12. *For $x, y \in \mathcal{B}$ let $A^x_y = \{z \in \mathcal{B} : \mu([z \succeq x]) \geq \frac{1}{2} \vee \mu([y \succeq z]) \geq \frac{1}{2}\}$. $\mu_{\mathcal{K}}$ is weakly coherent if $(\mu_{\mathcal{K}}([x \succeq y]) \geq \frac{1}{2}) \implies (\forall_{z \in A^x_y} : \mu_{\mathcal{K}}([x \succeq z \succeq y]) \geq$*

$\mu_{\mathcal{K}}([y \succeq z \succeq x]))$.

Proposition 4. *Expected preference $\omega_{\mathcal{K}}$ is transitive if and only if $\mu_{\mathcal{K}}$ is weakly coherent.*

Proof. I start with the first equivalence. Let $\mu_{\mathcal{K}}([x \succeq y]) \geq \frac{1}{2}, \mu_{\mathcal{K}}([y \succeq z]) \geq \frac{1}{2}$ and denote $A = \mu_{\mathcal{K}}([z \succeq x \succeq y])$, $B = \mu_{\mathcal{K}}([x \succeq z \succeq y])$, $C = \mu_{\mathcal{K}}([x \succeq y \succeq z])$, $D = \mu_{\mathcal{K}}([z \succeq y \succeq x])$, $E = \mu_{\mathcal{K}}([y \succeq z \succeq x])$, $F = \mu_{\mathcal{K}}([y \succeq x \succeq z])$. Now $x \succeq_{\mathcal{K}} y \iff A + B + C \geq D + E + F$, $y \succeq_{\mathcal{K}} z \iff C + E + F \geq A + B + D$ and $x \succeq_{\mathcal{K}} z \iff B + C + F \geq A + D + E$.

From assumption that $\mu_{\mathcal{K}}([x \succeq y]) \geq \frac{1}{2}$ and $\mu_{\mathcal{K}}([y \succeq z]) \geq \frac{1}{2}$ I get that $A + B + C \geq \frac{1}{2} \geq A + B + D$ so $C \geq D$. Now assume weak coherence holds. Applying it to $x \succeq_{\mathcal{K}} y$ I get $B \geq E$ and applying it to $y \succeq_{\mathcal{K}} z$ I get $F \geq A$. Therefore I get $B + C + F \geq A + D + E$ and $\omega_{\mathcal{K}}$ is transitive. Now assume $\omega_{\mathcal{K}}$ is transitive, so $B + C + F \geq A + D + E$ holds. Due to the assumption that $x \succeq_{\mathcal{K}} y$ I get

$$B + C + F \geq A + D + E \iff A + B + C + 2F \geq 2A + D + E + F \implies$$

$$\implies 2F \geq 2A \iff F \geq A.$$

Similarly from the assumption that $y \succeq_{\mathcal{K}} z$ I get

$$B + C + F \geq A + D + E \iff 2B + C + F + E \geq A + D + 2E + B \implies$$

$$\implies 2B \geq 2E \iff B \geq E,$$

so weak coherence holds.

□

By proposition 4 the transitivity of $\omega_{\mathcal{K}}$ is equivalent to weak coherence. As it turns out, the same property is sufficient for indirect rankings $\omega_{z_1}, \omega_{z_2}$ with reference points $z_1 \sim_{\mathcal{K}} z_2$ to coincide.

Corollary 2. *Assume $\mu_{\mathcal{K}}$ is weakly coherent. Then for every pair $z_1, z_2 \in \mathcal{N}$ such that $z_1 \sim_{\mathcal{K}} z_2$ I have $\omega_{z_1} = \omega_{z_2}$.*

Proof. Note, that for the case $z_1 \sim_{\mathcal{K}} z_2$ definitions 11 and 12 coincide, due to transitivity of $\omega_{\mathcal{K}}$ shown in proposition 4. Therefore by proposition 2 weak coherence of $\mu_{\mathcal{K}}$ implies that for any pair $z_1 \sim_{\mathcal{K}} z_2$ preference relations $\omega_{z_1}, \omega_{z_2}$ coincide. □

Weak coherence is a relaxation of coherence by restricting the domain of reference points for which it holds. In contrast to coherence, weak coherence allows for rankings induced by indirect comparisons to be reference point dependent, but not to an arbitrary degree. To illustrate this point further, consider the following example 3.

Example 3. Consider again the situation presented in the example 2. In this example, Bob considers "Titanic" to be the reference point, and gets the ranking of $u_T(G_2) > u_T(W) > u_T(T)$. However, were he to consider G as the reference point instead, his indirect preference ranking would be $u_G(W) > u_G(G) = u_G(G_2)$, meaning that his indirect preference between "Godzilla 2" and "Gone with the wind" is reference dependent.

As such, Bob's beliefs violate coherence. However, weak coherence is satisfied. His expected preference ranking is transitive and given by $W \succ G \sim G2 \succ T$.

I still have to show that weakly coherent measures exist. Moreover, natural questions remain whether I can represent every $\omega \in \Omega(\mathcal{K})$ in this way, and for a given $\omega \in \Omega(\mathcal{K})$ how to construct a measure that represents it. I answer these questions in theorem 2, which is the main result of this section. It is preceded by supplementary definitions 13-15 and lemma 4, which are only of technical importance.

Definition 13. *I denote by* $\mathrm{Diag}(\omega), \mathrm{Diag}^+(\omega), \mathrm{Diag}_-(\omega) \subset \mathcal{B} \times \mathcal{B}$ *sets of respectively diagonal, upper diagonal and lower diagonal elements of relation* ω*, that is*

$$\mathrm{Diag}(\omega) = \{(x,y) \in \mathcal{B} \times \mathcal{B} : x \sim_\omega y\},$$

$$\mathrm{Diag}^+(\omega) = \{(x,y) \in \mathcal{B} \times \mathcal{B} : x \succ_\omega y\},$$

$$\mathrm{Diag}_-(\omega) = \{(x,y) \in \mathcal{B} \times \mathcal{B} : x \prec_\omega y\}.$$

Definition 14. *Let* $\mu_\mathcal{K}$ *be given. I say that a measure* $\mu'_\mathcal{K}$ *is obtained from* $\mu_\mathcal{K}$ *by a disturbance* (μ', w') *if* μ' *is a probability measure defined on* $\Omega(\mathcal{K})$*, function* $w' : \mathcal{B}^2 \to [0,1]$ *satisfy* $w'(x,y) = w'(y,x)$ *and*

$$\mu'_\mathcal{K}([x \succ y]) = (1 - w'(x,y))\mu_\mathcal{K}([x \succ y]) + w'(x,y)\mu'([x \succ y]).$$

Definition 15. *Let* $\mu_\mathcal{K}$ *be given. The disturbance* (μ', w') *does not disturb the diagonal, if and only if for* $A = \mathrm{supp}(w')$ *I have*

1. $A \cap \mathrm{Diag}(\omega_{\mathcal{K}}) = \emptyset$,

2. $(x, y) \in A \cap \mathrm{Diag}^{+}(\omega_{\mathcal{K}}) \implies \mu'([x \succ y]) \geq \frac{1}{2}$, *with equality only for* $w'(x, y) < 1$,

3. $(x, y) \in A \cap \mathrm{Diag}_{-}(\omega_{\mathcal{K}}) \implies \mu'([x \succ y]) \leq \frac{1}{2}$, *with equality only for* $w'(x, y) < 1$.

If this is not the case, (μ', w') *disturbs the diagonal.*

Lemma 4. *Let* $\mu_{\mathcal{K}}$ *be given and* $\mu'_{\mathcal{K}}$ *be obtained from* $\mu_{\mathcal{K}}$ *by a disturbance* (μ', w') *that does not disturb the diagonal. Then* $\mu'_{\mathcal{K}}$ *also represents* $\omega_{\mathcal{K}}$.

Proof. Let $\mu'_{\mathcal{K}}$ be obtained from $\mu_{\mathcal{K}}$ without disturbing the diagonal and denote by $\omega'_{\mathcal{K}}$ (or $\succeq_{\mathcal{K}'}$) the relation given by definition 9 applied to $\mu'_{\mathcal{K}}$. Fix an arbitrary $x \in \mathcal{B}$. Following definition 15 I have $A \cap \{y \in \mathcal{B} : y \sim_{\mathcal{K}} x\} = \emptyset$. Therefore $x \sim_{\mathcal{K}} y \iff x \sim_{\mathcal{K}'} y$. Now let $y \in \mathcal{B}$ be such that $y \succ_{\mathcal{K}} x$. If $(y, x) \notin \mathrm{supp}(w')$ then obviously $y \succ_{\mathcal{K}'} x$, so assume that $(y, x) \in \mathrm{supp}(w')$. Now by definition of a disturbance

$$\mu'_{\mathcal{K}}([y \succ x]) = (1 - w'(y, x))\mu_{\mathcal{K}}([y \succ x]) + w'(y, x)\mu'([y \succ x]).$$

By assumption $y \succ_{\mathcal{K}} x$ I have $\mu_{\mathcal{D}}([y \succ x]) > \frac{1}{2}$. Moreover following definition 15 I have $\mu'([y \succ x]) \geq \frac{1}{2}$. Therefore $\mu'_{\mathcal{K}}([y \succ x]) > \frac{1}{2}$ and $y \succ_{\mathcal{K}'} x$. As the case with $x \succ_{\mathcal{K}} y$ is symmetric to this one, $\omega_{\mathcal{K}} = \omega'_{\mathcal{K}}$ and therefore $\mu'_{\mathcal{K}}$ also represents $\omega_{\mathcal{K}}$.

□

Theorem 2. *Let* $\mathcal{K}$ *be fixed. For any given* $\omega \in \Omega(\mathcal{K})$ *there exists some weakly coherent measure* $\mu_{\mathcal{K}}$ *representing* ω. *Moreover, for some fixed* $\omega_{\mathcal{K}} \in \Omega(\mathcal{K})$, *there is a* $\mu_{\mathcal{K}}$ *representing* $\omega_{\mathcal{K}}$ *that can be represented as*

$$\mu_{\mathcal{K}}([x \succeq y]) = \sum_{i=1}^{n} w_i(x, y)\mu_i([x \succ y]) + (1 - \sum_{i=1}^{n} w_i(x, y))\mu_*([x \succeq y]),$$

where (μ_i, w_i) *are disturbances that do not disturb the diagonal that for all* $x, y \in \mathcal{B}$ *satisfy* $\sum_{i=1}^{n} w_i(x, y) \leq 1$ *and* μ_* *is a coherent measure on* Ω *representing* $\omega_{\mathcal{K}}$.

Proof. Due to corollary 1, I can restrict my attention only to values of $\mu_{\mathcal{K}}$ on the sets $[x \succ y]$. As $\omega_{\mathcal{K}} \in \Omega(\mathcal{K})$ there is a continuous utility function that represents it. Let u be this utility function, and denote by x^*, y_* some maximum and minimum elements for relation $\omega_{\mathcal{K}}$. As $\mathcal{B}$ is compact and $\omega_{\mathcal{K}}$ is continuous, such x^*, y_* exist.

Define $\mu_*([x \succ y]) = \frac{1}{2} + \frac{u(x)-u(y)}{2(u(x^*)-u(y_*))}$. Clearly $\mu_*([x \succeq y]) \geq \frac{1}{2} \iff u(x) \geq u(y)$, and therefore it represents $\omega_{\mathcal{K}}$ on Ω. Moreover, for any $z \in \mathcal{B}$ I have $\mu_*([x \succeq z]) \geq \mu_*([y \succeq z]) \iff u(x) \geq u(y)$ and therefore μ_* is coherent. However, it cannot represent $\omega_{\mathcal{K}}$ on $\Omega(\mathcal{K})$ as it is not restricted to $\Omega(\mathcal{K})$, so for $d_1 \succ d_2 \in \mathcal{K}$ does not imply $\mu_*([d_1 \succ d_2]) = 1$ unless $d_1 \sim_{\mathcal{K}} x^*$ and $d_2 \sim_{\mathcal{K}} y_*$. Note however, that from definition 9 I have $d_1 \succ d_2 \in \mathcal{K} \iff d_1 \succ_{\mathcal{K}} d_2$, and therefore $\mu_*([d_1 \succ d_2]) > \frac{1}{2}$.

By lemma 4, if I disturb μ_* without disturbing the diagonal, the disturbed measure also represents $\omega_{\mathcal{K}}$. I now show that there is a sequence $(\mu_i, w_i)_{i=1}^n$ of disturbances that does not disturb the diagonal, such that $(1-\sum_{i=1}^n w_i(x,y))\mu_*([x \succ y]) + \sum_{i=1}^n w(x,y)\mu_i([x \succ y])$ is equal to 0 whenever $y \succeq x \in \mathcal{K}$. I can assume without loss of generality that $\mathcal{K}$ consists of strict preference relations only and I denote all known relations as $\mathcal{K} = \{x_i \succ y_i : i \leq n\}$.

For all i, fix some pairwise disjoint $B_i = B((x_i, y_i), r_i) \subset \text{Diag}^+(\omega_{\mathcal{K}})$ and define

$$w_i(x,y) = \max\left\{1 - \frac{d((x,y),(x_i,y_i))}{r_i}, 1 - \frac{d((x,y),(y_i,x_i))}{r_i}, 0\right\},$$

$$u_i(x) = \begin{cases} 1 & \text{if } u(x) > u(x_i), \\ 0 & \text{if } u(x) < u(y_i), \\ \frac{u(x)-u(y_i)}{u(x_i)-u(y_i)} & \text{otherwise.} \end{cases}$$

It suffices to take $\mu_i([x \succ y]) = \frac{1}{2} + \frac{u(x)+u(y)}{2}$. By construction each disturbance (μ_i, w_i) does not disturb the diagonal and as a result $\mu_{\mathcal{K}}([x \succ y]) = (1-\sum_{i=1}^n w_i(x,y))\mu_*([x \succ y]) + \sum_{i=1}^n w(x,y)\mu_i([x \succ y])$ represents $\omega_{\mathcal{K}}$. Moreover $w_i(x_i, y_i) = 1$ and $\mu_i([x_i \succ y_i]) = 1$, so μ is restricted to $\Omega(\mathcal{K})$. As additionally $\mu_{\mathcal{K}}$ is of the requested form, the proof is finished.

□

Although the main point of theorem 2 is the existence of weakly coherent measures, this result says a lot more than that. It tells that any $\omega \in \Omega(\mathcal{K})$ can be represented by some $\mu_{\mathcal{K}}$, therefore showing again the generality of the proposed language. This result also gives a functional form for a measure $\mu_{\mathcal{K}}$ that represents a given $\omega \in \Omega(\mathcal{K})$. This representation is conceptually similar to the one obtained by Gilboa and Schmeidler (1995)[6], as $\mu_{\mathcal{K}}([x \succ y])$ is a weighted average of the values assigned to the measures that are conditional on each relation in $\mathcal{K}$ separately.

[6]In order to see this similarity, note that each $x_i \succ y_i \in \mathcal{K}$ can be interpreted as a known "case" and w_i as constructed in the proof of theorem 2 is monotone with respect to similarity.

This is not a unique representation, as I can clearly add another disturbance than does not disturb the diagonal if I wish. However, this representation is especially important, in the sense that it does not disturb more than needed — this point is made precise by corollary 3.

Corollary 3. *Let μ_* be a coherent measure on Ω representing $\omega_{\mathcal{K}}$ for some $\mathcal{K} = \{x_1 \succ \cdots \succ x_n\}$, $n \geq 3$ and $\mu_{\mathcal{K}}$ be a measure on $\Omega(\mathcal{K})$ obtained from μ_* by a disturbance (μ', w') that does not disturb the diagonal.*

1. *Let $j \in \{2, \ldots, n-1\}$ and $i \in \{1, \ldots, n\}$. Then $w'(x_i, x_j) = 1$.*
2. *Let U be an arbitrary open subset of $\mathcal{B}^2$ such that $cp(\mathcal{K}) \implies (x, y) \in U$. There exists (μ', w') such that $\operatorname{supp}(w') \subset U$ and μ_* disturbed by (μ', w') represents ω on $\Omega(\mathcal{K})$.*

Proof. Follows straight from construction of $\mu_{\mathcal{K}}$ in the proof of theorem 2.

□

This corollary tells us firstly that any coherent measure μ_* must be disturbed in every[7] pair in $\mathcal{B}^2$ for which the relation is known and secondly that those points are the only ones in which the disturbance is really necessary in order to obtain $\mu_{\mathcal{K}}$ that represents $\omega_{\mathcal{K}}$ on $\Omega(\mathcal{K})$ from some coherent μ_* on Ω. Of course, continuity means, that the disturbance must spill over to some neighbourhood of those points.

6 Final notes

In this article I have presented a cognitive basis for the formation of preferences of a taste uncertain consumer. My main contribution is the construction of probability measures on the space of all preference relations that are intrinsically connected to the information available to the consumer. In this way I provide a general language for a formal study of a taste uncertain consumer. My results, most notably theorem 1, allow for an easy and interpretable definition of such a measure. This result is also empirically significant, as it specifies precisely the data needed to estimate this measure.

[7] Unless the best known alternative x_1 and the worst known x_n are respectively maximal and minimal elements of relation $\omega_{\mathcal{K}}$. In this case I do not have to disturb the neighborhood of pairs $(x_1, x_n), (x_n, x_1) \in \mathcal{B}^2$.

I have shown how to identify expected preference and risk perception of the taste uncertain consumer, using respectively direct and indirect comparisons. Since risk perception in my model is inherently reference dependent, my results give a cognitive justification for the formation of reference dependent preferences. Finally, under the additional assumption of weak coherence, I have shown in theorem 2 not only that my formulation of expected preference is very general and each permissible preference relation can be represented in this way, but also that there is a very natural form of the measure that represents this preferences.

At the same time, I am yet to consider either the experimental behavior of the consumer, or the behavioural implications of the model. I do so briefly in the remainder of this section.

Experimentation and learning

My construction is restrictive from the perspective of the study of learning. Firstly, I assume that consumption perfectly reveals preference rankings between alternatives, so direct learning from consumption is trivial. Secondly, the construction provided by theorem 1 does not restrict indirect learning in any way beside the demand for continuity of μ. Thirdly and most importantly, by definition 2 the consumer perfectly anticipates changes in $\mu_{\mathcal{K}}$ that result from a new information. As a consequence, in order to study learning of the consumer in any significant detail, extension of my results to a richer setting is required.

Much of the discussion above applies to the study of other dynamic properties, including experimentation. At the same time, a brief discussion of experimentation is possible. As experimentation demands considering alternative sets of known alternatives $\mathcal{D}$, in order to simplify the notation in this discussion, I denote $\mathcal{K}$ as $\mathcal{K}(\mathcal{D})$, to signify the dependence. Definition 16 formalizes the notion of experimental preferences, which I denote by ω_E.

Definition 16. *Let $x, y \in \mathcal{B}$ and assume $\mathcal{D}$ is given. Denote $\mathcal{D}_x = \mathcal{D} \cup \{x\}$ and $\mathcal{D}_y = \mathcal{D} \cup \{y\}$. I say that x is experimentally preferred to y, denoted by $x \succ_E y$ if $E_{\mu_{\mathcal{K}(\mathcal{D})}}[\mu(\Omega(\mathcal{K}(\mathcal{D}_x)))] < E_{\mu_{\mathcal{K}(\mathcal{D})}}[\mu(\Omega(\mathcal{K}(\mathcal{D}_y)))]$.*

In order to understand what is going on in the definition 16, consider that $\mu(\Omega(\mathcal{K}(\mathcal{D})))$ can be seen as a natural measure of taste uncertainty that is yet to be resolved. Therefore it feels natural to consider x as resolving more uncertainty then y if $\mu(\Omega(\mathcal{K}(\mathcal{D}_x))) < \mu(\Omega(\mathcal{K}(\mathcal{D}_y)))$. However, both of those values are ex ante unknown, as they depend on what the revealed relations between x, y and the elements of $\mathcal{D}$ will turn out to be. Therefore ex ante, $\mu(\Omega(\mathcal{K}(\mathcal{D}_x)))$ and $\mu(\Omega(\mathcal{K}(\mathcal{D}_y)))$ are both random variables and definition 16 considers that x is experimentally preferred to y if the expected value of the random variable $\mu(\Omega(\mathcal{K}(\mathcal{D}_x)))$ is lower than that of $\mu(\Omega(\mathcal{K}(\mathcal{D}_y)))$.

Thanks to the definition 2, I can easily obtain a representation for ω_E.

Proposition 5. *Experimental preferences ω_E are always complete, transitive, continuous and reflexive. Moreover let $\mathcal{K} = \{x_1 \succeq \cdots \succeq x_n\}$. Then the utility function $u_E(x) = 1 - \sum_{i=1}^{n-1} \mu^2_{\mathcal{K}}([x_i \succ x \succ x_{i+1}]) - \mu^2_{\mathcal{K}}([x \succ x_1]) - \mu^2_{\mathcal{K}\sqcup}([x_n \succ x])$ represents ω_E.*

Proof. Since u_E as defined in the statement of the theorem is continuous, the second part of the theorem implies the first part. Therefore I only need to prove that u_E represents ω_E. From definition 2 I have the following

$$E_{\mu_{\mathcal{K}(\mathcal{D})}}\left[\mu(\Omega(\mathcal{K}(\mathcal{D}_x)))\right] = \mu([\mathcal{K}(\mathcal{D}) \cup \{x_n \succ x\}])\mu_{\mathcal{K}(\mathcal{D})}([x_n \succ x]) +$$

$$+\mu([\mathcal{K}(\mathcal{D}) \cup \{x \succ x_1\}])\mu_{\mathcal{K}(\mathcal{D})}([x \succ x_1]) + \sum_{i=1}^{n-1} \mu([\mathcal{K}(\mathcal{D}) \cup \{x_i \succ x, x \succ x_i+1\}])\mu_{\mathcal{K}(\mathcal{D})}([x_i \succ x \succ x_{i+1}]) =$$

$$= \mu^2_{\mathcal{K}(\mathcal{D})}([x_n \succ x])\mu(\Omega(\mathcal{K}(\mathcal{D}))) + \mu^2_{\mathcal{K}(\mathcal{D})}([x \succ x_1])\mu(\Omega(\mathcal{K}(\mathcal{D}))) +$$

$$+ \sum_{i=1}^{n-1} \mu^2_{\mathcal{K}(\mathcal{D})}([x_i \succ x \succ x_{i+1}])\mu(\Omega(\mathcal{K}(\mathcal{D}))),$$

and therefore

$$x \succeq_E y \iff (1 - u_E(x))\mu(\Omega(\mathcal{K}(\mathcal{D}))) \leq (1 - u_E(y))\mu(\Omega(\mathcal{K}(\mathcal{D}))) \iff u_E(x) \geq u_E(y).$$

□

Proposition 5 gives a very natural utility function for experimental preferences. Let $\mathcal{K} = \{x_1 \succ \cdots \succ x_n\}$ and denote probability of x being in i-th position in the ranking of known alternatives as p_i, meaning that $p_i = \mu_{\mathcal{K}}([x_{i-1} \succ x \succ x_{i+1}])$ for $i = 2, \ldots, n$, with $p_1 = \mu_{\mathcal{K}}([x \succ x_1])$ and $p_{n+1} = \mu_{\mathcal{K}}([x_n \succ x])$. Then $1 - u_E$ is simply a quadratic form $\sum_{i=1}^{n+1} p_i^2$ and for example the maximal element with respect to ω_E is the one such that $p_i = \frac{1}{n+1}$ for all i (if such an element exists), meaning that each position in the resulting preference ranking is equally probable.

Preference reversal

The observation of Cox and Grether (1996) that the preference reversal paradox, first reported by Lichtenstein and Slovic (1971), is less prevalent in a setting with repeated choices and incentives to experiment, was the main motivation behind the formulation of the discovered preferences hypothesis by Plott (1996). However, the two explanations of this paradox that are most widely accepted in the literature do not explain why this should be the case. Those two are firstly the explanation of Tversky et al. (1988) that choice and valuations tasks employ different decision modes, meaning that the attributes are weighted differently between those tasks; and secondly the explanation of Sugden (2003) that the preference reversal can result from a very small variations of the reference points. I do not propose an alternative explanation. Instead, my aim is to integrate existing explanations with the preference discovery.

I fix my attention on the preference reversal reported by Stalmeier et al. (1997). Let (x, y) denote living for x years with migraine for y days a week (followed by death). Stalmeier et al. (1997) observe, that most subjects prefer $(10, 5)$ to $(20, 5)$, but at the same time they evaluated $(20, 5)$ to be equivalent to a longer period of life in good health than $(10, 5)$. It is safe to assume that the preferences of the subjects are monotone with respect to the period for which they are alive — longer life is preferred as long as the health state is preferred to death and vice versa. As such, the explanation of Tversky et al. (1988) is not viable in this case. I focus therefore on the explanation in the spirit of Sugden (2003).

I fix a reference dependent utility function[8] to be $u_z(x) = \mu_{\mathcal{K}}([x \succ z])$. I assume that in the choice task, the subjects evaluate both alternatives using the other as a reference point, meaning that the preference for a shorter life takes the form of $u_{(20,5)}(10, 5) > u_{(10,5)}(20, 5)$. In other words, direct comparison is used. The observation of Bostic et al. (1990) suggests that in the matching task[9] the subjects employ a different reference point, which I take to be their current health

[8]It does not have to be a pure indirect utility function, but it allows the exposition to be clearer.

[9]Bostic et al. (1990) report that the ratio of the observed preference reversals decreases significantly if instead of asking the subjects in the matching task to report the value for which they are indifferent between this value and an alternative, the subject are given a sequence of pairwise choices between the values and the alternative in question.

state, denoted A. Therefore the matching task reveals that $u_A(10,5) < u_A(20,5)$.

From proposition 2 I know that u_z is reference dependent as long as it is not coherent. From proposition 3 it is clear that I can assume coherence is not satisfied for the subjects. However, if $\mu_{\mathcal{K}}$ is weakly coherent, the reversal described above is possible only if firstly, A is either strictly preferred to both $(10,5)$ and $(20,5)$ or both those alternatives are strictly preferred to A; secondly, if the subjects are uncertain in their preference for at least one of $(10,5),(20,5)$. As the subjects in the experiment of Stalmeier et al. (1997) were high school students with the mean age of 17, I can safely assume that their current health state and life expectancy is preferred to both the alternatives, and that they have never experienced a prolonged period of migraine, so that there is some taste uncertainty present. As such, the proposed theory can explain this case of the preference reversal.

My model allows for a more detailed explanation. Denote having a migraine for 5 days a week by M and death by D. By the monotonicity assumption, $(20,5)$ is preferred if $M \succ_{\omega^*} D$, and $(10,5)$ in the other case. Therefore, $(20,5)$ is a more risky prospect. From the choice task it is clear that $\mu_{\mathcal{K}}([D \succ M]) = p > \frac{1}{2}$. In order to understand the preference for $(20,5)$ in the matching task, consider that A is strictly better to both of the prospects in question in terms of expected preference. However, A is uncertain. The subjects are probably aware, that there is a non-zero probability that in a few days they will be diagnosed with a terrible illness. Therefore $\mu_{\mathcal{K}}([(x,5) \succ A]) > 0$. Since in the case if $M \succ_{\omega^*} D$ $(20,5)$ is better to $(10,5)$ it is not surprising that $(20,5)$ can be seen as having a higher probability out of those two of being at least as good as A.

7 References

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