STUDIA EKONOMICZNE ● ECONOMIC STUDIES
NR 3 (XC) 2016

ARTYKUŁY DYSKUSYJNE

Leszek Zaremba[*]

REPLICATION OF FINANCIAL INSTRUMENTS WHICH COMPENSATE LOWER SHARE PRICES

(Artykuł nadesłany: 31.08.2016; Zaakceptowany: 16.02.2017)

ABSTRACT

In this paper our main goal is to demonstrate how listed companies can successfully defend themselves against falling share prices. We present a one-period model of the Polish financial market from the view point of KGHM. The ideas, notions, and tools presented in this article have high potential to be useful also to investment funds because the so-called replicating portfolios, when properly chosen, have negative prices and generate positive or zero income in all scenarios (states of the market). Financial instruments are represented here by vectors, while financial markets, by matrices. Having in mind that the stock price of KGHM declined from 126 PLN on April 15, 2015 to 57.50 PLN on January 15, 2016 and stayed at this level until May 15, 2016, we show how KGHM could create a financial instrument (with negative cost) which would fully compensate big potential declines of its share prices.

Keywords: replication error, hedging, approximate hedging, expected sum of squared replication errors, incomplete market, risk management.

JEL Classification: C02, C18, C54, C60

[*] Institute of Finance, Vistula University, ul. Stokłosy 3. 02-787 Warszawa, e-mail: l.zaremba@vistula.edu.pl.

1. INTRODUCTION

In this paper we address an important problem concerning potential declines in share prices of listed companies. Our key goal is to show how such companies can successfully defend themselves against such events. They should build up a portfolio (which costs nothing or even has a negative price) consisting of long and short positions of certain securities. Such portfolio will in almost 100% compensate even big declines of their share prices. The motivation for writing this article came from the observation that the Polish company KGHM, one of the largest cooper and silver producers in the world, suffered huge declines in share prices. In fact, the stock price of KGHM declined from 126.45 PLN on April 15, 2015 to 57.50 PLN on January 15, 2015, and stayed at this or slightly higher level until May 15, 2016 when it reached the price of 60.31 PLN[1].

We present a one-period model of a financial market in which there are only two dates, say today and tomorrow, or equivalently this week and the next week, etc. The most important feature of this model is that no economic activity (consumption, trading and work) is carried out in between the two dates because all activity takes place solely "today" and "tomorrow".

Despite these simplifications, this model quite adequately represents the real financial market from the point of view of companies which try to manage their risks associated with uncertain share prices. It is also adequate for investment funds which, by means of the methodology presented here, will be able to generate nonnegative cash flows with practically no cost.

It is well known (Cerny, 2009) that each vector (written typically in a column form) such as for example $\mathbf{b} = \begin{bmatrix} 135 \\ 110 \\ 85 \\ 65 \end{bmatrix}$, features pay-offs resulting from a given security, e.g. a company's share, while matrix, such as

$$\mathbf{A} = \begin{bmatrix} 135 & 100 & 45 & 0 \\ 110 & 100 & 20 & 0 \\ 85 & 100 & 0 & 5 \\ 65 & 100 & 0 & 25 \end{bmatrix}, \tag{1}$$

represents a financial market, for example the Polish financial market, with $\mathbf{b}$ showing payments resulting from one share of KGHM. Thus, this simplified market model consists of just four liquid securities by means of which one can create other securities. The third column shows payments resulting from a call option of one KGHM share at the strike price 90 PLN, while the fourth column features payments resulting from a put option of one KGHM share, also at the

[1] pl.investing.com/equities/kghm-polska-miedz-sa-historical-data

strike price 90 PLN. The four rows of matrix A refer to four possible states (scenarios) of the Polish financial market.

2. PROBLEM STATEMENT

In this paper we shall study a slightly more sophisticated one-period matrix model

$$P = \begin{bmatrix} 70 & 100 & 0 \\ 85 & 100 & 0 \\ 100 & 100 & 5 \\ 115 & 100 & 20 \\ 125 & 100 & 30 \end{bmatrix}, \tag{2}$$

of the Polish financial market than that given by matrix A. Suppose that today is April 15, 2015 and one share of KGHM costs 126 PLN. Our investment horizon is 9 months. The three prices significantly lower than 126 PLN (i.e. 100, 85, 70) in the first column indicate that a hypothetical financial engineer hired by KGHM in August 2015 presumed that the share price of KGHM might decline in a 9-month period down to 70 PLN or even further. The first column represents thus payments in January 2016 resulting from a single share of KGHM in all five different states (scenarios) of the Polish market. The remaining two columns also feature payments in February 2016 resulting from two other securities.

Namely, the second column represents a treasury bill paying 100 PLN in all five states, while the third column shows payments resulting from a call option of one share of KGHM at exercise (strike) price 95 PLN.

One of the most natural and conceptually simple ways to counteract big potential declines in share prices of any company has been to buy, say, a put option with strike price equal to the current share price. In case of company KGHM, we might buy a 9-month put option with strike price 120 PLN. Such option would generate the pay-offs

$$\mathbf{d} = \begin{bmatrix} 50 \\ 35 \\ 20 \\ 5 \\ 0 \end{bmatrix} \tag{3}$$

in five states which would fully compensate potential declines in KGHM share prices. An important factor that should be taken into account is the price (premium) of such an option.

In this paper we offer an even more general approach than that focusing on replicating the put option mentioned above because we will be looking for the *best approximate hedge* for any security similar to $\mathbf{d}$, say $\mathbf{d}'$, which will not only compensate declines in share prices, but will cost less. This can be done the following way:

In the first part of this paper we will replicate the instrument **d** and then $\mathbf{d}' = \begin{bmatrix} 53 \\ 36 \\ 19 \\ 3 \\ 0 \end{bmatrix}$.

It will appear that **d'** is cheaper than d; in fact both prices are negative, but the portfolio replicating financial instrument **d'** pays even more than the portfolio replicating **d**. To make our model more realistic, we associate certain probabilities to 5 scenarios (states) of the Polish financial market. They can be given by the vector

$$\mathbf{p} = \begin{bmatrix} 0.075 \\ 0.125 \\ 0.210 \\ 0.400 \\ 0.190 \end{bmatrix}, \tag{4}$$

assigning smaller probabilities to deeper declines (scenarios one and two) of KGHM share prices in a 9-month period than to slight decreases (scenarios three and four). The choice of probability vector **p** is somewhat arbitrary and does not affect our reasoning and methodology.

3. THEORY

Suppose a financial market under consideration is represented by a matrix (see: Cerny, 2009)

$$\mathrm{A} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{5}$$

is a desired instrument for purchase by the company ABC. The question arises, can this instrument be replicated by means of the liquid securities (columns) of matrix A, and with what level of accuracy.

Definition 1. If each vector (financial instrument) can be replicated perfectly (without any error) by means of columns of matrix A, then we shall say that the financial market represented by A is *complete*. Otherwise, we shall say that the financial market is *incomplete*.

It is well known (Cerny, 2009) that a financial market is complete only if the matrix-vector equation $A \cdot x = b$ has a solution for each vector **b** and the resulting hedging portfolio is then given by the formula $x = A^{-1}b$. Complete markets from a theoretical point of view were presented recently in Zaremba (2015). In this paper we are dealing with incomplete markets, both from theoretical and, primarily, practical point of view.

However, if the financial market is incomplete ($A \cdot x = b$ does not have a solution), then the question arises of how one can build the *best approximate hedge* of instrument **b**. In other words, the company desiring to possess instrument **b** should look for such portfolio $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, consisting of columns of matrix A (liquid securities on the financial market under consideration) which recreates **b** in the best possible way, that is, with the smallest replication error

$$\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_m) = \mathrm{Ax} - \mathrm{b} \tag{6}$$

between coordinates of replica Ax and vector **b** in a sense of the sum of squared replication errors, SSRE), where

$$\mathrm{SSRE} = \varepsilon_1{}^2 + \varepsilon_2{}^2 + ... + \varepsilon_m{}^2 = \left[(Ax)_1 - b_1\right]^2 + \left[(Ax)_2 - b_2\right]^2 + ... + \left[(Ax)_m - b_m\right]^2 \tag{7}$$

is as small as possible. Here $(\mathrm{Ax})_i$ denotes the "i"-th coordinate of vector Ax, while b_i stands for the "i"-th coordinate of vector **b**. Let us note that formula (7) assigns equal weights to each market scenario (state).

In reality, however, some states of the world are less likely than others and consequently the company should be interested rather in expected SSRE than in SSRE, where

$$\begin{aligned} \mathrm{ESSRE} &= p_1\varepsilon_1{}^2 + p_2\varepsilon_2{}^2 + ... + p_m\varepsilon_m{}^2 = \\ &= p_1\left[(Ax)_1 - b_1\right]^2 + p_2\left[(Ax)_2 - b_2\right]^2 + ... + p_m\left[(Ax)_m - b_m\right]^2 \end{aligned} \tag{8}$$

with $p_1 > 0, p_2 > 0, \ldots, p_m > 0$ denoting objective probabilities of the individual states of the world; m denotes the numbers of rows (scenarios) that may take place in our model.

The following result can be found in Cerny (2009):

Fact 1. Consider a general hedging problem $Ax = b$ having in mind the corresponding replication error (6). Define a new matrix $\tilde{A}$ and a new vector $\tilde{b}$ by multiplying each row of A and **b** by the square root of the probability for the corresponding state. The optimal hedging portfolio that minimizes ESSRE is of the form $\hat{x} = [\tilde{A}^T \tilde{A}]^{-1} \tilde{A}^T \tilde{b}$. Its payments are given by vector $A\hat{x} = A[\tilde{A}^T \tilde{A}]^{-1} \tilde{A}^T \tilde{b}$ which in the best possible way replicates the desired **b**.

4. SOLUTION TO THE HEDGING PROBLEM

This strong and important result has many practical applications. For example, let the role of A be played by matrix P representing the Polish financial market (see (2)), with vector **b** = **d** given by (3), representing the security desired by KGHM. Let us compute for KGHM the best replicating portfolio $\hat{x} = [\tilde{P}^T \tilde{P}]^{-1} \tilde{P}^T \tilde{d}$, and then the resulting pay-off.

$$\tilde{P} = \begin{bmatrix} 70\cdot\sqrt{0.075} & 100\cdot\sqrt{0.075} & 0\cdot\sqrt{0.075} \\ 85\cdot\sqrt{0.125} & 100\cdot\sqrt{0.125} & 0\cdot\sqrt{0.125} \\ 100\cdot\sqrt{0.21} & 100\cdot\sqrt{0.21} & 5\cdot\sqrt{0.21} \\ 115\cdot\sqrt{0.40} & 100\cdot\sqrt{0.40} & 20\cdot\sqrt{0.40} \\ 125\cdot\sqrt{0.19} & 100\cdot\sqrt{0.19} & 35\cdot\sqrt{0.19} \end{bmatrix} = \begin{bmatrix} 19.17 & 27.39 & 0 \\ 30.05 & 35.36 & 0 \\ 45.83 & 45.83 & 2.29 \\ 72.73 & 63.25 & 12.65 \\ 54.49 & 43.59 & 13.08 \end{bmatrix} \quad (9)$$

hence

$$(\tilde{P})^T \tilde{P} = \begin{bmatrix} 11629 & 10663 & 1737.5 \\ 10663 & 10000 & 1475 \\ 1737.5 & 1475 & 336.25 \end{bmatrix} \text{ and } [(\tilde{P})^T \tilde{P}]^{-1} = \begin{bmatrix} 0.0320 & -0.027 & -0.0438 \\ -0.0300 & 0.024 & 0.0365 \\ -0.0428 & 0.036 & 0.0692 \end{bmatrix}. \quad (10)$$

Since $(\tilde{P})^T \tilde{d} = \begin{bmatrix} 1284.4 \\ 1432.5 \\ 61.0 \end{bmatrix}$, the resulting optimal portfolio is

$$\hat{x} = \begin{bmatrix} 0.032 & -0.027 & -0.0438 \\ -0.03 & 0.024 & 0.0365 \\ -0.0438 & 0.036 & 0.0692 \end{bmatrix} \cdot \begin{bmatrix} 1284.4 \\ 1432.5 \\ 61.0 \end{bmatrix} = \begin{bmatrix} -1.084 \\ 1.264 \\ 0.238 \end{bmatrix},$$

and it generates the pay-offs

$$P\cdot\hat{x} = \begin{bmatrix} 70 & 100 & 0 \\ 85 & 100 & 0 \\ 100 & 100 & 5 \\ 115 & 100 & 20 \\ 125 & 100 & 30 \end{bmatrix} \begin{bmatrix} -1.084 \\ 1.264 \\ 0.238 \end{bmatrix} = \begin{bmatrix} 50.50 \\ 4.25 \\ 19.18 \\ 6.50 \\ -1.95 \end{bmatrix} \neq \begin{bmatrix} 50 \\ 35 \\ 20 \\ 5 \\ 0 \end{bmatrix} \quad (11)$$

which only slightly differ from the desired ones given by (3), and shown again in the last column of (11). The corresponding expected SSRE, given by formula (see (8))

$$\text{ESSRE} = p_1\left[(Ax)_1 - b_1\right]^2 + p_2\left[(Ax)_2 - b_2\right]^2 + \ldots + p_m\left[(Ax)_m - b_m\right]^2$$

is really small in the studied case, namely

$$\text{ESSRE} = 0.075\left[0.5\right]^2 + 0.125\left[0.75\right]^2 + 0.21[0.82]^2 + 0.4\left[1.5]\right]^2 + 0.19[1.95]^2 = 1.856.$$

It is worth mentioning that the best approximate hedge $\hat{x}$ is offering 50.50 PLN in the worst case scenario one, which is even more than the desired by KGHM instrument **d** is promising. A similar situation takes place in the most likely scenario four, where the replication portfolio $\hat{x}$ is paying more (6.50 PLN) than KGHM requested (5 PLN).

5. THE COST OF THE BEST APPROXIMATE HEDGING MAY BE CLOSE TO ZERO

The question that arises is, what should be the cost assigned to portfolio $\hat{x} = \begin{bmatrix} -1.084 \\ 1.264 \\ 0.238 \end{bmatrix}$? Since this portfolio is built of three securities whose pay-offs are featured by three columns of matrix $\mathrm{P} = \begin{bmatrix} 70 & 100 & 0 \\ 85 & 100 & 0 \\ 100 & 100 & 5 \\ 115 & 100 & 20 \\ 125 & 100 & 30 \end{bmatrix}$, our question can be reformulated as follows: what are the market prices of one share of KGHM company, one treasury bill paying 100 PLN 9 months from now, and the call option with strike 95 PLN (represented by the third column), call it **f**. Since we have assumed that the current market price of KGHM share is 126 PLN, and the current price of a 9-month treasury bill must be around 98.50 PLN, the only thing we need to discuss is the possible market price of the call option **f**.

Instead of invoking the Black-Scholes formula to valuate call option f, let us make some general observations which will give us enough insight to compute the cost of portfolio $\hat{x}$. Firstly, the maximal price of **f** must be lower than 30 PLN because its market price is a discounted value of its pay-offs in five different scenarios (states) with the maximal payment (in scenario five) being equal to 30 PLN. Therefore, the cost of portfolio $\hat{x}$ must be significantly less than

$$-1.084 \cdot 126 + 1.264 \cdot 98.5 + 0.238 \cdot 30 = -4.94 \text{ PLN}, \tag{12}$$

which means that it is negative!

Let us now compute the expected payment resulting from instrument **f**. Using the vector-matrix convention, one can express the expected payment as

$$< f^T; p >= \begin{bmatrix} 0 & 0 & 5 & 20 & 30 \end{bmatrix} \begin{bmatrix} 0.075 \\ 0.125 \\ 0.21 \\ 0.40 \\ 0.19 \end{bmatrix} = 14.75 \text{ PLN}, \tag{13}$$

that is, far less than 30 PLN. Assume, however, that the price of **f** is between 30 and 14.75, say 25 PLN. Then the corresponding price of portfolio $\hat{x}$ would be equal to

$$< S^T ; \hat{x} >= \begin{bmatrix} 126 \\ 98.5 \\ 25 \end{bmatrix}^T \cdot \begin{bmatrix} -1.084 \\ 1.264 \\ 0.238 \end{bmatrix} = -6.13 . \tag{14}$$

In this way we have proved

Corollary 1. The cost of the best approximate portfolio $\hat{x} = \begin{bmatrix} -1.084 \\ 1.264 \\ 0.238 \end{bmatrix}$ yielding payoffs $\begin{bmatrix} 50.50 \\ 34.25 \\ 19.18 \\ 6.50 \\ -1.95 \end{bmatrix}$ instead of desired ones $\begin{bmatrix} 50 \\ 35 \\ 20 \\ 5 \\ 0 \end{bmatrix}$ is negative. What's more, the holder of such portfolio is paid between 4.94 PLN and 6.13 PLN.

6. THE COST OF THE BEST APPROXIMATE HEDGING MAY BE NEGATIVE

Let us start with a simple observation.

Remark 1. If the share price of KGHM did not fell from 126 PLN in April 2015 but stayed steady until January 2016 or even rised above 126 PLN, then the decision to hold portfolio $\hat{x} = \begin{bmatrix} -1.084 \\ 1.264 \\ 0.238 \end{bmatrix}$, would also be very fortunate for KGHM, not because portfolio $\hat{x}$ was able to compensate declines in KGHM share prices (as, according to our assumption, they did not fall), but because of its negative purchasing price and practically nonnegative cash flow it generates, as shown by vector $\begin{bmatrix} 50.50 \\ 34.25 \\ 19.18 \\ 6.50 \\ -1.95 \end{bmatrix}$. Fortunately, the negative "payment" of –1.95 PLN can be easily absorbed by the high price of KGHM's shares(125 PLN). Concluding this remark, let us notice that buying at negative price 100 million of such portfolios $\hat{x}$ when call options cost 25 PLN each, would yield an extra 613 million PLN to KGHM because

$$< S^T ; \hat{x} >= \begin{bmatrix} 126 \\ 98.5 \\ 25 \end{bmatrix}^T \cdot \begin{bmatrix} -1.084 \\ 1.264 \\ 0.238 \end{bmatrix} = -6.13. \tag{15}$$

Now we will show how one can build a replication portfolio which pays even more than portfolio $\hat{x} = \begin{bmatrix} -1.084 \\ 1.264 \\ 0.238 \end{bmatrix}$ does. Toward this end, let the role of financial instrument $\mathbf{d} = \begin{bmatrix} 50 \\ 35 \\ 20 \\ 5 \\ 0 \end{bmatrix}$ be played by instrument $\mathbf{d}' = \begin{bmatrix} 53 \\ 36 \\ 19 \\ 3 \\ 0 \end{bmatrix}$. Proceeding in exactly the same way as above, we obtain the same matrices $(\tilde{P})^T \tilde{P}$ and $[(\tilde{P})^T \tilde{P}]^{-1}$ (see (10)), but different vector $(\tilde{P})^T \tilde{d}' = \begin{bmatrix} 1197.8 \\ 1366.5 \\ 44.0 \end{bmatrix}$, what will result in the portfolio $\hat{x}' = \begin{bmatrix} -1.276 \\ 1.432 \\ 0.444 \end{bmatrix}$, different from $\hat{x}$, and with pay-offs $\begin{bmatrix} 53.86 \\ 34.72 \\ 17.79 \\ 5.31 \\ -3.02 \end{bmatrix}$ which even better than payments $\begin{bmatrix} 50 \\ 35 \\ 20 \\ 5 \\ 0 \end{bmatrix}$ compensate the decline of share prices in the worst case scenario one (53.86 > 50). However, the most beneficial feature of portfolio $\hat{x}' = \begin{bmatrix} -1.276 \\ 1.432 \\ 0.444 \end{bmatrix}$ is its price. When $\mathbf{f} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 20 \\ 30 \end{bmatrix}$ costs 25 PLN, then the cost of $\hat{x}'$ will be equal to $-1.276 \cdot 126 + + 1.432 \cdot 98.5 + 0.444 \cdot 25 = -8.62$ PLN. Therefore, “buying” 100 million of such portfolios, would yield an extra 249 million PLN to KGHM (249 = 862 – 613).

7. CONCLUDING REMARKS

It was shown that potential declines of share prices could be successfully compensated by appropriately chosen portfolios. The choice of KGHM was quite random what means that one might replace KGHM with any other listed company in Poland or abroad and proceed basically in the same way to generate a positive or almost positive cash flow by means of a portfolio whose cost is negative.

This author personally believes that adding more basis assets to the matrix representing a given financial market, such as call options and put options with various striking prices and the same expiration date, would be beneficial for the model employed here. For example, matrix with five columns and five rows (scenarios) would be a good start for further expansion of this model.

REFERENCES

Cerny A. (2009), *Mathematical Techniques in Finance*, Princeton University Press.

Zaremba L. (2015), *Inżynieria finansowa na rynkach zupełnych*, „Kwartalnik Naukowy Uniwersytetu Vistula", 46(4), s. 101–115.

REPLIKOWANIE INSTRUMENTÓW FINANSOWYCH REKOMPENSUJĄCYCH SPADKI KURSÓW AKCJI NA GIEŁDZIE

STRESZCZENIE

Celem artykułu jest pokazanie na przykładzie KGHM, jak spółki giełdowe w Polsce i na świecie mogą rekompensować sobie spadki cen ich akcji na giełdzie, tworząc odpowiedni portfel. W artykule zaprezentowano 1-okresowy model, który przedstawia polski rynek finansowy z punktu widzenia interesów firmy KGHM. Zaprezentowane tu idee, pojęcia i narzędzia mogą być również użyteczne dla wszelkiego typu funduszy inwestycyjnych. Instrumenty finansowe pokazane są w tym artykule jako wektory, zaś rynki finansowe jako macierze. Biorąc pod uwagę to, iż kurs KGHM spadł z poziomu 126 zł w dniu 15.04.2015 do 57,50 zł w dniu 15.01.2016 i pozostawał na tym poziomie aż do maja 2016 r., pokazano w tym artykule, w jaki sposób KGHM mógł zreplikować sobie za darmo i w bardzo precyzyjny sposób taki instrument finansowy (zdefiniowany w tej publikacji), który w pełni rekompensuje nawet duże spadki kursu giełdowego KGHM.

Słowa kluczowe: błąd replikacyjny, hedging, aproksymacja hedgingu, oczekiwana suma kwadratów błędów replikacyjnych, rynek niezupełny, zarządzanie ryzykiem.

Klasyfikacja JEL: C02, C18, C54, C60