International sentiment spillovers (preliminary results)*

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Abstract

To Be Added

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Extended abstract

Pigou (1927) and Keynes (1936) believed that business cycles are driven by animal spirits. However, in the macroeconomic literature there is no agreement whether they are indeed. Blanchard et al. (2013), Angeletos and La'O (2013) suggest that beliefs (noise or sentiment shocks) massively contribute to output, consumption and investment fluctuations in the United States. Others claim that this impact is rather small (Barsky and Sims (2012)).

These studies were conducted in closed-economy setting. Our paper extends the topic asking whether beliefs are a local phenomenon or rather spill over across economies. If they do, they might help explain the comovement in main macro variables that are often observed between large economies and small countries that are strongly integrated with each other.

To address this question, we construct a two-country New Keynesian DSGE model that accounts for noise shocks in the spirit of Blanchard et al. (2013) in both economies. In the model agents receive signals about long-lasting productivity changes in domestic and foreign countries. Agents have to disentangle whether signals reflect real productivity changes or noise. The latter, even though it is a non-fundamental shock, generates comovement of output, consumption and investment and thus resembles Keynes' animal spirits. Otherwise the framework is standard, two-country New Keynesian DSGE model with capital adjustment costs, variable capital utilization, sticky prices and wages, local currency pricing and conventional monetary policy of Taylor type (see Appendix for more details).

We estimate the model with US and Canadian data using 13 time series: productivity, consumption, investments, wages, inflation and nominal short term interest rates for both economies plus the exchange rate. We include 19 shocks in the estimation, calibrate some parameters and estimate others.

As impulse responses point out, the US noise shock leads to synchronization of macroeconomic variables in both economies. Furhtermore, noise shock can be associated with some important events such as oil shocks, change in the US monetary policy or financial crisis. Such events impact agents' beliefs about long-term economic growth and in this way they help in explaining the comovement between consumption and GDP in both economies. In line with Blanchard et al. (2013) we show that in the US a significant share (around 30%) of consumption volatility can be explained by US noise shock. This shock turns out to spill over to the small economy (Canada) and explain around 15% of consumption volatility in this economy. As a consequence beliefs help explain the comovement between the two economies.

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Tables and Figures

Table 1: Variance decomposition: consumption growth in the US

Quarter	CAN pp	CAN tp	US pp	US tp	CAN noise	US noise
1	0.0	2.06	0.5	26.1	0.0	37.6
4	0.0	1.9	8.2	27.4	0.0	25.9
8	0.0	1.7	18.5	21.7	0.0	26.4
12	0.0	1.6	20.1	19.1	0.0	23.4
40	0.0	1.5	20.9	17.7	0.0	21.9

Table 2: Variance decomposition: consumption in Canada

Quarter	CAN pp	CAN tp	US pp	US tp	CAN noise	US noise
1	0.0	16.5	90.4	4.4	0.0	16.9
4	0.0	14.2	4.2	4.2	0.0	15.0
8	0.0	12.0	7.4	3.5	0.0	14.5
12	0.0	11.4	7.8	3.3	0.0	13.8
40	0.0	11.1	8.1	3.4	0.0	13.3

Table 3: Variance decomposition: GDP in the US

Quarter	CAN pp	CAN tp	US pp	US tp	CAN noise	US noise
1	0.0	0.0	0.1	16.1	0.0	6.0
4	0.0	0.0	0.9	14.4	0.0	5.8
8	0.0	0.1	2.6	14.1	0.0	5.6
12	0.0	0.1	3.4	13.9	0.0	5.5
40	0.0	0.1	4.3	13.6	0.0	5.4

Table 4: Variance decomposition: GDP in Canada

Quarter	CAN pp	CAN tp	US pp	US tp	CAN noise	US noise
1	0.0	32.5	0.1	10.0	0.0	1.7
4	0.0	29.1	0.9	9.7	0.0	1.8
8	0.0	26.1	2.0	8.8	0.0	1.9
12	0.0	25.6	2.5	8.6	0.0	1.9
40	0.0	25.2	3.2	8.4	0.0	1.9

Table 5: Foreign sentiment shock: impact on correlations

correlation (CAN,US)	without noise shock	with noise shock
GDP	0.30	0.41
consumption	0.07	0.09

Table 6: Estimated parameters

	prior mean	post. mean	90% HPD	interval	prior	pstdev
hh	0.500	0.5845	0.5682	0.6012	beta	0.1000
hh_s	0.500	0.7595	0.7322	0.7861	beta	0.1000
cap_theta	5.000	5.1772	4.4045	5.9554	norm	0.5000
cap_theta_s	5.000	4.8040	3.9390	5.6351	norm	0.5000
gamma_u2	0.150	0.0679	0.0237	0.1084	beta	0.0500
gamma_u2_s	0.150	0.1518	0.1406	0.1626	beta	0.0500
gam_r	0.700	0.8612	0.8464	0.8770	beta	0.1000
gam_pic	0.100	0.1057	0.0830	0.1261	beta	0.0500
gam_y	0.100	0.1969	0.1646	0.2234	beta	0.0500
gam_r_s	0.700	0.8016	0.7772	0.8270	beta	0.1000
gam_pic_s	0.100	0.0748	0.0519	0.0945	beta	0.0500
gam_y_s	0.100	0.0133	0.0056	0.0205	beta	0.0500
lambda_x	0.800	0.9556	0.9208	0.9916	beta	0.1000
thetaH	0.750	0.7253	0.6567	0.7834	beta	0.1000
thetaF	0.750	0.9806	0.9718	0.9891	beta	0.1000
thetaH_s	0.750	0.4650	0.4261	0.5100	beta	0.1000
thetaF_s	0.750	0.8827	0.8365	0.9350	beta	0.1000
zetaH	0.750	0.7428	0.6743	0.8073	beta	0.1000
zetaF	0.750	0.7448	0.6520	0.8329	beta	0.1000
zetaH_s	0.750	0.6657	0.6274	0.7057	beta	0.1000
zetaF_s	0.750	0.8141	0.7507	0.8646	beta	0.1000
thetaW	0.750	0.9592	0.9567	0.9611	beta	0.0500
zetaW	0.750	0.5608	0.5083	0.6113	beta	0.1000
thetaW_s	0.750	0.9571	0.9429	0.9716	beta	0.1000
zetaW_s	0.750	0.6267	0.5723	0.6950	beta	0.1000
theta_muH_lag	0.500	0.1203	0.0727	0.1737	beta	0.1000
theta_muH_s_lag	0.500	0.1126	0.0903	0.1408	beta	0.1000
theta_muW_lag	0.500	0.6895	0.6175	0.7630	beta	0.1000
theta_muW_s_lag	0.500	0.8172	0.7691	0.8640	beta	0.1000
rho_x	0.900	0.9448	0.9277	0.9599	beta	0.0500
rho_x_s	0.900	0.9678	0.9578	0.9774	beta	0.0500
rho_i	0.700	0.4604	0.4381	0.4876	beta	0.0500
rho_i_s	0.700	0.4085	0.3808	0.4317	beta	0.0500
rho_muH	0.700	0.5455	0.5128	0.5743	beta	0.0500
rho_muH_s	0.700	0.5604	0.5350	0.5852	beta	0.0500
rho_muW	0.700	0.8626	0.8441	0.8839	beta	0.0500
rho_muW_s	0.700	0.7591	0.7323	0.7871	beta	0.0500
rho_rho	0.700	0.9196	0.9082	0.9313	beta	0.0500
sig_x	0.005	0.0151	0.0141	0.0162	invg	0.0010
sig_x_s	0.005	0.0235	0.0230	0.0239	invg	0.0010
sig_s	0.010	0.0063	0.0032	0.0095	invg	0.0100
sig_s_s	0.010	0.0094	0.0054	0.0141	invg	0.0100
sig_r	0.001	0.0024	0.0022	0.0027	invg	Inf
sig_r_s	0.001	0.0023	0.0021	0.0025	invg	Inf
sig_i	0.010	0.0915	0.0823	0.1003	invg	Inf
sig_i_s	0.010	0.2075	0.1879	0.2280	invg	Inf
sig_muH	0.010	0.0268	0.0192	0.0330	invg	Inf
sig_muH_s	0.010	0.0230	0.0142	0.0324	invg	Inf
sig_muW	0.010	0.0825	0.0684	0.0984	invg	Inf
sig_muW_s	0.010	0.0080	0.0025	0.0144	invg	Inf
sig_rho	0.010	0.0032	0.0027	0.0037	invg	Inf
sig_c_ME	0.001	0.0008	0.0004	0.0013	invg	0.0010
sig_c_ME_s	0.001	0.0037	0.0033	0.0040	invg	0.0010
sig_i_ME	0.001	0.0002	0.0001	0.0003	invg	0.0010
sig_i_ME_s	0.001	0.0012	0.0004	0.0017	invg	0.0010
sig_w_ME	0.001	0.0093	0.0088	0.0097	invg	0.0010
sig_w_ME_s	0.001	0.0081	0.0077	0.0084	invg	0.0010

Figure 1: The impulse responses to foreign sentiment shock.

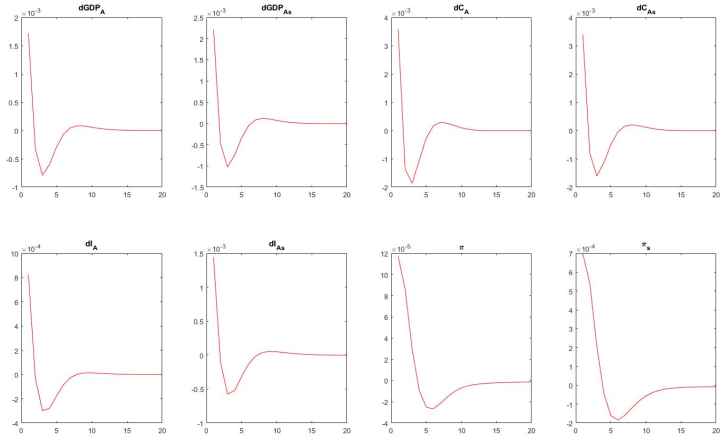


Figure 2: Foreign sentiment shock: smoothed path.

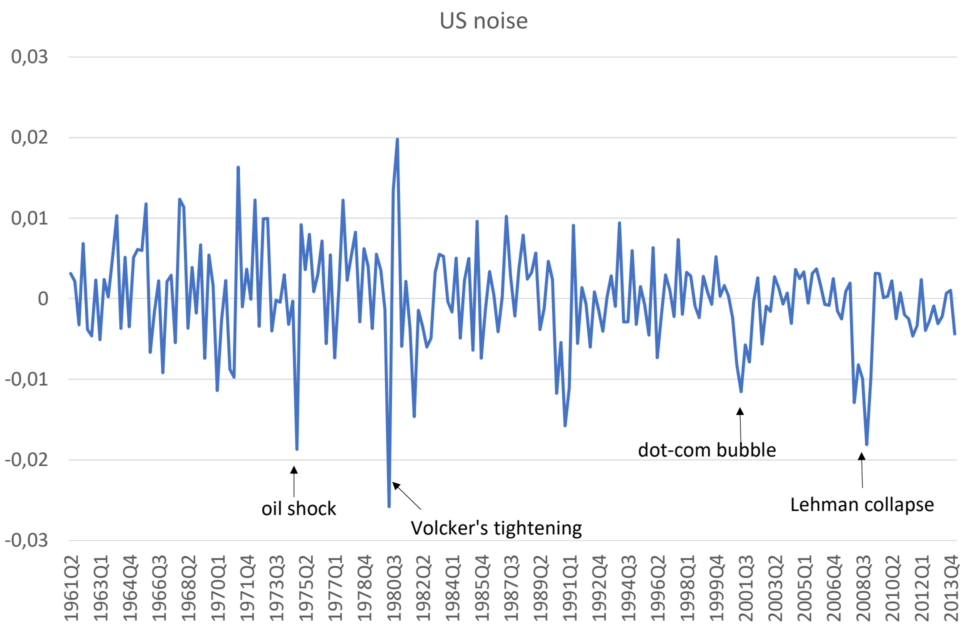
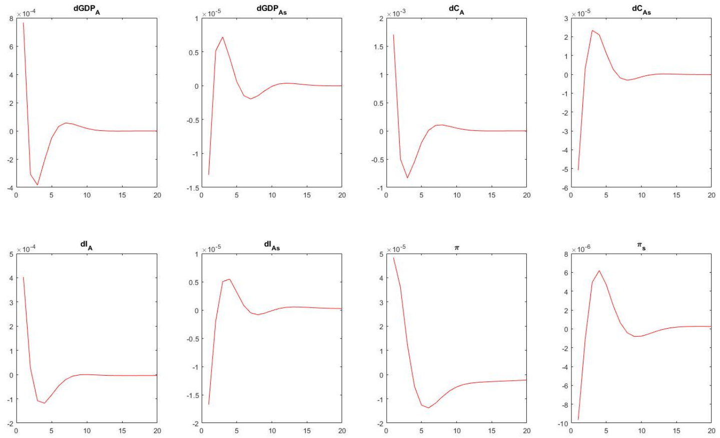


Figure 3: The impulse responses to domestic sentiment shock



Appendix. Model

This appendix presents derivations of a two-country dynamic stochastic general equilibrium model for the project on sentiment spillovers. The size of economies are: ω and $1-\omega$. The model accounts for sentiment shocks in both economies and otherwise represents a standard New Keynesian open macroeconomic framework. The sentiment shock comes from uncertainty faced by agents who receive noisy signals about permanent productivity changes. Below, we present the model setup.

1 Households

Households in both economies face symetric problem. They maximise their lifetime utility U_0 that is a function of their consumption c_t and household labour supply n_t:

$$\begin{aligned} U_0 &= E_0 \sum_{t=0}^{\infty} exp(\varepsilon_t^d)\beta^t u(c_t, n_t) \\ &= E_0 \sum_{t=0}^{\infty} exp(\varepsilon_t^d)\beta^t \left[\log(c_t - hc_{t-1}) - \gamma \frac{1}{1+\varphi} n_t^{1+\varphi}\right] \end{aligned}$$

where E_0 denotes agents' expectations at the time 0, ε_t^d - a time preference shock, β - a discount rate, $u()$ - a period utility function, while γ and φ are standard parameters. The utility maximization is subject to household budget constraint (in real terms):

$$c_t + b_{H,t} + q_t b_{F,t} + t_t = R_{t-1}\pi_t^{-1} b_{H,t-1} + q_t R_{t-1}^* \Gamma_{t-1} \pi_t^{*-1} b_{F,t-1} + w_t^h n_t + d_t + d_t^c + d_t^L$$

where $b_{H,t}$ stands for bond holdings issued by the domestic government in the domestic currency, q_t - the real exchange rate vis-a-vis the foreign economy, $b_{F,t}$ - bond holdings issued by the foreign government in the foreign currency, t_t - lump-sum transfers from the domestic government, R_{t-1} - nominal domestic interest rate (from period t-1 to period t), π_t - domestic (consumer) inflation rate (from period t-1 to period t), R_{t-1}^* - nominal foreigninterest rate (from period t-1 to period t), Γ_{t-1} - transaction cost on holding foreign bonds, π_t^* - foreign (consumer) inflation rate (from period t-1 to period t), w_t^h - real wage, d_t- profits paid out by intermediate firms, d_t^c - profits paid out by capital producers, d_t^L- profits paid out by labour aggregators.

First order conditions (FOC) of household maximisation problem are as follows (where $\Lambda_t \equiv u_{c,t}$ is Lagrange multiplier on budget constraint):

$$[c_t] : \Lambda_t = exp(\varepsilon_t^d)(c_t - hc_{t-1})^{-1} - \beta h(c_{t+1} - hc_t)^{-1}$$

$$[n_t] : w_t = \Lambda_t \gamma n_t^{\varphi}$$

$$[b_{H,t}] : \Lambda_t = \beta E_t \left\{ \Lambda_{t+1} R_t \pi_{t+1}^{-1} \right\}$$

$$[b_{F,t}] : \Lambda_t q_t = \beta E_t \left\{ \Lambda_{t+1} q_{t+1} \Gamma_t R_t^* \pi_{t+1}^{*-1} \right\}$$

1.1 Labour market

We assume that households have a continuum of labour services of measure one $i \in [0,1]$ that they sell to labour agencies at the nominal price $W_t = P_t w_t$. Labour servies are marked by labour agents so that they become imperfect substitutes. Therefore, the agencies receive heterogenous wages $W_t(i)$ that are set according to the Calvo scheme with wage stickiness parameter equal θ_W. Labour is aggregated by a perfectly competetive labour aggregator who combines all labour types $n_t(i)$ into a homogenous labour service n_t according to the following formula:

$$n_t = \left(\int_0^1 n_t(i)^{\frac{1}{1+\mu_w}} di \right)^{1+\mu_w}$$

We assume that if a labour type is not allowed to optimise its wage it indexes it:

$$w_{t+1}(i) = \pi_t^{\zeta,w} w_t(i)$$

where:

$$\pi_t^{\zeta,w} = (1 - \zeta_W)\bar{\pi} + \zeta_W \pi_{t-1}^W$$

If the labour agency is allowed to reset the price, it solves the following problem:

$$\max_{\tilde{w}_t(i), \{n_t(i)\}_{s=0}^{\infty}} E_t \sum_s (\beta\theta_W)^s \lambda_{t,t+s} \left(\tilde{w}_t(i) \pi_{t,t+s}^{\zeta,w} - w_{t+s}^h \right) n_{t+s}(i)$$

subject to the demand of labour aggregators:

$$n_{t+s}(i) = \left(\frac{\tilde{w}_t(i) \pi_{t+s}^{\zeta,w}}{w_{t+s}} \right)^{\frac{\mu_w}{1-\mu_w}} n_{t+s}$$

where $\pi_{t,t+s}^{\zeta,w} = \pi_{t+1}^{\zeta,w} \cdot \ldots \cdot \pi_{t+s}^{\zeta,w}$. Solving this problem we get:

$$\tilde{w}_t = \mu_w \frac{\Omega_{w,t}}{\Upsilon_{w,t}}$$

where:

$$\Omega_{w,t} = \lambda_t w_t^h w_t^{\frac{\mu_w}{\mu_w - 1}} n_t + \beta\theta_w E_t \left(\frac{\pi_{t+1}^{\zeta,w}}{\pi_{t+1}} \right)^{\frac{\mu_w}{1-\mu_w}} \Omega_{w,t+1}$$

and

$$\Upsilon_{w,t} = \lambda_t w_t^{\frac{\mu_w}{\mu_w - 1}} n_t + \beta\theta_w E_t \left(\frac{\pi_{t+1}^{\zeta,w}}{\pi_{t+1}} \right)^{\frac{1}{1-\mu_w}} \Upsilon_{w,t+1}$$

2 Firms

We consider saveral stages of prodcution process. In each economy capital producers use undepreciated capital from the previous period and investments as inputs in producing capital that is used by retailers to produce domestic and exporting goods. Final goods producers, in turn, combine goods sold by domestic and foreign retailers into final goods that are subsequently used for consumption and investments.

2.1 Capital producers

Competetive capital producers decide on investments and rent capital to intermediate good producing firms in order to maximize:

$$\max_{\{I_t\}_{t=0}^{\infty}} E_0 \sum_t \Lambda_t \beta^t \left(k_t \frac{R_t^k}{P_t} - i_t - C(u_t) \bar{k}_{t-1} \right)$$

subject to:

$$\bar{k}_t = (1-\delta)\bar{k}_{t-1} + \varepsilon_t^i \left(1 - S\left(\frac{i_t}{i_{t-1}} \right) \right) i_t$$

$$k_t = u_t \bar{k}_{t-1}$$

$$C(u_t) = u_t^{1+\zeta_u}/(1+\zeta_u)$$

$$\frac{dC(u_t)}{du_t} = u_t^{\zeta_u}$$

alternative (a la Huertgen or NAWM):

$$k_t = u_t \bar{k_{t-1}}$$

$$C(u_t) = \gamma_{u1}(u_t - 1) + \frac{\gamma_{u2}}{2}(u_t - 1)^2$$

$$\frac{dC(u_t)}{du_t} = \gamma_{u1} + \gamma_{u2}(u_t - 1)$$

where k_t denotes effective capital, u_t - its utilization rate, ϵ_t^i - investment shock, $C(u_t)$ - cost of capital utilization, R_t^k - return from renting capital to retailers, $S(\cdot)$ stands for the investment adjustment costs ($S'(\cdot) > 0$ and $S''(\cdot) > 0$). We assume: $S\left(\frac{i_t}{i_{t-1}}\right) = \frac{\Theta}{2}\left(\frac{i_t}{i_{t-1}} - 1\right)^2$. Note that $S' = \frac{\partial S\left(\frac{i_t}{i_{t-1}}\right)}{\partial i_t} = \frac{\Theta}{i_{t-1}}\left(\frac{i_t}{i_{t-1}} - 1\right)$ and $S(1) = S'(1) = 0$.

The Lagrange function is given by:

$$L = E_0 \left\{ \sum_t \Lambda_t \beta^t \left(u_t \bar{k_{t-1}} \frac{R_t^k}{P_t} - i_t - C(u_t)\bar{k_{t-1}} \right) + \lambda_{k,t} \left[(1-\delta)\bar{k_{t-1}} + \varepsilon_t^i \left(1 - S\left(\frac{i_t}{i_{t-1}}\right) \right) i_t - \bar{k}_t \right] \right\}$$

FOC are as follows:

$$[i_t] : E_0 \left\{ -\Lambda_t \beta^t + \lambda_{k,t} \varepsilon_t^i \left[1 - S\left(\frac{i_t}{i_{t-1}}\right) - i_t S'\left(\frac{i_t}{i_{t-1}}\right) \right] \right\} = 0$$

$$[\bar{k_{t-1}}] : E_0 \left\{ \Lambda_t \beta^t u_t \left(\frac{R_t^k}{P_t}\right) - \Lambda_t \beta^t C(u_t) + \lambda_{k,t}(1-\delta) - \lambda_{k,t-1} \right\} = 0$$

$$[u_t] : E_0 \left\{ \Lambda_t \beta^t \bar{k_{t-1}} \left(\frac{R_t^k}{P_t}\right) - \Lambda_t \beta^t \frac{dC(u_t)}{du_t} \bar{k_{t-1}} \right\} = 0$$

From the above equations we get up to the first order:

$$\Lambda_{t+1}\beta u_{t+1}\left(\frac{R_{t+1}^k}{P_{t+1}}\right) - \Lambda_{t+1}\beta C(u_{t+1}) - \frac{\Lambda_t}{\varepsilon_t^i \left\{ 1 - S\left(\frac{i_t}{i_{t-1}}\right) - i_t S'\left(\frac{i_t}{i_{t-1}}\right) \right\}} + \\ + \frac{\Lambda_{t+1}\beta}{\varepsilon_{t+1}^i \left(1 - S\left(\frac{i_{t+1}}{i_t}\right) - i_{t+1} S'\left(\frac{i_{t+1}}{i_t}\right) \right)}(1-\delta) = 0$$

$$\frac{R_t^k}{P_t} = \frac{dC(u_t)}{du_t}$$

Profits of capital producers are given by:

$$d_t^c = \frac{R_t^k k_t}{P_t} - i_t - C(u_t)\bar{k}_{t-1}$$

2.2 Final good producers

Final good producers act in the perfectly competetive market. They combine domestic and foreign output into homogenous goods that are used for consumption and investment. Thus, the domestic producer maximizes profits given by

$$P_t y_t - P_{H,t} y_{H,t} - P_{F,t} y_{F,t}$$

subject to the following production technology

$$y_t = \left[\eta^{\frac{\mu-1}{\mu}} (y_{H,t})^{\frac{1}{\mu}} + (1-\eta)^{\frac{\mu-1}{\mu}} (y_{F,t})^{\frac{1}{\mu}}\right]^{\mu}$$

FOC:

$$\begin{aligned}
P_{F,t} &= \lambda_t \mu \left((1-\eta)^{\frac{\mu-1}{\mu}} (y_{F,t})^{\frac{1}{\mu}} + \eta^{\frac{\mu-1}{\mu}} (y_{H,t})^{\frac{1}{\mu}}\right)^{\mu-1} \frac{1}{\mu} (1-\eta)^{\frac{\mu-1}{\mu}} (y_{F,t})^{\frac{1-\mu}{\mu}} \\
P_{H,t} &= \lambda_t \mu \left((1-\eta)^{\frac{\mu-1}{\mu}} (y_{F,t})^{\frac{1}{\mu}} + \eta^{\frac{\mu-1}{\mu}} (y_{H,t})^{\frac{1}{\mu}}\right)^{\mu-1} \frac{1}{\mu} \eta^{\frac{\mu-1}{\mu}} (y_{F,t})^{\frac{1-\mu}{\mu}}
\end{aligned}$$

Demands Simplifying:

$$\begin{aligned}
\left(\frac{P_{F,t}}{P_t}\right)^{\frac{\mu}{\mu-1}} &= \left((1-\eta)^{\frac{\mu-1}{\mu}} (y_{F,t})^{\frac{1}{\mu}} + \eta^{\frac{\mu-1}{\mu}} (y_{H,t})^{\frac{1}{\mu}}\right)^{\mu} (1-\eta)(y_{F,t})^{-1} \\
\left(\frac{P_{H,t}}{P_t}\right)^{\frac{\mu}{\mu-1}} &= \left((1-\eta)^{\frac{\mu-1}{\mu}} (y_{F,t})^{\frac{1}{\mu}} + \eta^{\frac{\mu-1}{\mu}} (y_{H,t})^{\frac{1}{\mu}}\right)^{\mu} \eta(y_{H,t})^{-1}
\end{aligned}$$

$$\begin{aligned}
y_{F,t} &= (1-\eta)\,(p_{F,t})^{\frac{\mu}{1-\mu}}\, y_t \\
y_{H,t} &= \eta\,(p_{H,t})^{\frac{\mu}{1-\mu}}\, y_t
\end{aligned}$$

Analogously the demands for $y_{H,t}^*$ and $y_{F,t}^*$ are given by the following

$$\begin{aligned}
y_{H,t}^* &= \eta^* \left(p_{H,t}^*\right)^{\frac{\mu^*}{1-\mu^*}} y_t^* \\
y_{F,t}^* &= (1-\eta^*) \left(p_{F,t}^*\right)^{\frac{\mu^*}{1-\mu^*}} y_t^*
\end{aligned}$$

Inflation Zero profit condition:

$$y_t = \left[\eta^{\frac{\mu-1}{\mu}} \left(\eta \left(\frac{P_{H,t}}{P_t} \right)^{\frac{\mu}{1-\mu}} y_t \right)^{\frac{1}{\mu}} + (1-\eta)^{\frac{\mu-1}{\mu}} \left((1-\eta) \left(\frac{P_{F,t}}{P_t} \right)^{\frac{\mu}{1-\mu}} y_t \right)^{\frac{1}{\mu}} \right]^{\mu}$$

$$y_t = \left[\eta \left(\frac{P_{H,t}}{P_t} \right)^{\frac{1}{1-\mu}} y_t^{\frac{1}{\mu}} + (1-\eta) \left(\frac{P_{F,t}}{P_t} \right)^{\frac{1}{1-\mu}} y_t^{\frac{1}{\mu}} \right]^{\mu}$$

$$P_t^{\frac{\mu}{1-\mu}} = \left((1-\eta) \left(P_{F,t}\right)^{\frac{1}{1-\mu}} + \eta \left(P_{H,t}\right)^{\frac{1}{1-\mu}} \right)^{\mu}$$

Analogously:

$$P_t^{*\frac{\mu^*}{1-\mu^*}} = \left((1-\eta^*) \left(P^*_{F,t}\right)^{\frac{1}{1-\mu^*}} + \eta^* \left(P^*_{H,t}\right)^{\frac{1}{1-\mu^*}} \right)^{\mu^*}$$

2.3 Domestic retailers

There are two kinds of intermediate goods producers in each economy: domestic goods producers and exporters. Domestic firms are assumed to utilize a standard Cobb-Douglas production function:

$$y_{p,t}(i) = k_t(i)^{\alpha} (A_t n_t(i))^{1-\alpha} - \phi$$

where ϕ corresponds to the fixed cost of production that guarantees that economic profits are rougly equal to zero in the steady state. A_t is a labour-augmenting technology process that we describe in more details in the next paragraph.

Technology Technology is the product of two components, permanent X_t and transitory Z_t:

$$A_t = \Gamma^t X_t Z_t$$

where Γ denotes the steady state TFP growth rate.

The permanent component follows a unit root process:

$$\frac{X_t}{X_{t-1}} = \left(\frac{X_{t-1}}{X_{t-2}} \right)^{\rho_X} exp(\varepsilon_{X,t})$$

The stationary component follows an AR(1) process:

$$Z_t = Z_{t-1}^{\rho_Z} exp(\varepsilon_{Z,t})$$

Agents observe A_t and a noisy signal S_t about its unit root component:

$$S_t = X_t V_t$$

where $\varepsilon_{X,t}$. $\varepsilon_{Z,t}$ and V_t are normal i.i.d. noise shocks. V_t is interpreted as the sentiment shock. We describe the filtering problem of agents in section 6.

Cost minimization Retailers act in monopolistically competetive market, hence they solve the cost minimization problem. (the problem is the same for all firms, thus we omit the subscript i):

$$TC = \frac{R_t^k}{P_t} k_t + w_t n_t$$

subject to:

$$y_{p,t} = {k_t}^{\alpha} (A_t n_t)^{1-\alpha} - \phi$$

This gives us FOC (ϑ_t stands for the Langrange multiplier):

$$\frac{R_t^k}{P_t} + \vartheta_t MPK_t = 0$$

$$w_t + \vartheta_t MPL_t = 0$$

thus:

$$\frac{R_t^k}{P_t w_t} = \frac{MPK_t}{MPL_t}$$

and:

$$\frac{R_t^k}{P_t w_t} = \frac{\alpha}{1-\alpha} \frac{n_t}{k_t}$$

Marginal cost We substitute the last equation into the production function:

$$y_{p,t} = {k_t}^{\alpha} (A_t \frac{1-\alpha}{\alpha} k_t \frac{R_t^k}{P_t w_t})^{1-\alpha} - \phi = k_t (\frac{1-\alpha}{\alpha} A_t \frac{R_t^k}{P_t w_t})^{1-\alpha} - \phi$$

and into total costs:

$$TC = \frac{R_t^k}{P_t} k_t + \frac{1-\alpha}{\alpha} k_t \frac{R_t^k}{P_t} = \frac{1}{\alpha} k_t \frac{R_t^k}{P_t}$$

since

$$k_t = (\frac{1-\alpha}{\alpha} A_t \frac{R_t^k}{P_t w_t})^{\alpha-1} (y_{p,t} + \phi)$$

we get:

$$TC_t = \frac{1}{\alpha}\frac{R_t^k}{P_t}(\frac{1-\alpha}{\alpha}A_t\frac{R_t^k}{P_t w_t})^{\alpha-1}(y_{p,t}+\phi)$$

and

$$MC_t = p_{m,t} = \frac{1}{\alpha}\frac{R_t^k}{P_t}(\frac{1-\alpha}{\alpha}A_t\frac{R_t^k}{P_t w_t})^{\alpha-1}$$

Price setting We apply buyer's currency pricing. Domestic firms set their price $\tilde{P}_t(i)$ to maximize:

$$\max_{\tilde{P}_t(i),\{y_t(i)\}_{s=0}^{\infty}} E_t \sum_s (\beta\theta_H)^s \Lambda_{t,t+s}\left(\frac{\tilde{P}_{H,t}(i)\,\pi_{t,t+s}^{\zeta}}{P_{t+s}} - p_{m,t+s}\right) y_{H,t+s}(i)$$

subject to the demand of final domestic good producers:

$$y_{H,t+s}(i) = \left(\frac{\tilde{P}_{H,t}(i)\,\pi_{t+s}^{\zeta}}{P_{H,t+s}}\right)^{\frac{\mu_H}{1-\mu_H}} y_{H,t+s}$$

where $\pi_{t,t+s}^{\zeta} = \pi_{t+1}^{\zeta}\cdot\ldots\cdot\pi_{t+s}^{\zeta}$. Solving this problem we get:

$$E_t\left[\sum_s (\beta\theta_H)^s \Lambda_{t,t+s}\left(\frac{\frac{1}{1-\mu_H}\left(\tilde{P}_{H,t}(i)\,\pi_{t,t+s}^{\zeta}\right)^{\frac{\mu_H}{1-\mu_H}}\pi_{t,t+s}^{\zeta}}{P_{t+s}} - \frac{\mu_H}{1-\mu_H}\left(\tilde{P}_{H,t}(i)\,\pi_{t,t+s}^{\zeta}\right)^{\frac{\mu_H}{1-\mu_H}-1}\pi_{t,t+s}^{\zeta}p_{m,t+s}\right)\left(\frac{1}{P_{H,t+s}}\right)^{\frac{\mu_H}{1-\mu_H}} y_{H,t+s}\right] = 0$$

and finally:

$$\tilde{p}_t = \mu_H \frac{E_t\sum_s (\beta\theta_H)^s \lambda_{t+s}\left(\frac{\pi_{t,t+s}^{\zeta}}{\pi_{t,t+s}}\right)^{\frac{\mu_H}{1-\mu_H}} p_{m,t+s}\, p_{H,t+s}^{\frac{\mu_H}{\mu_H-1}} y_{H,t+s}}{E_t\sum_s (\beta\theta_H)^s \tau_{H,t+s}\lambda_{t+s}\left(\frac{\pi_{t,t+s}^{\zeta}}{\pi_{t,t+s}}\right)^{\frac{1}{1-\mu_H}} p_{H,t+s}^{\frac{\mu_H}{\mu_H-1}} y_{H,t+s}}$$

where: $\Lambda_{t,t+1} = \frac{\lambda_{t+1}}{\lambda_t}$.

The expression above can be rewritten as:

$$\tilde{p}_{H,t} = \mu_H \frac{\Omega_{H,t}}{\Upsilon_{H,t}}$$

where:

$$\Omega_{H,t} = \lambda_t p_{m,t} p_{H,t}^{\frac{\mu_H}{\mu_H - 1}} y_{H,t} + \beta\theta_H E_t \left(\frac{\pi_{t+1}^{\zeta}}{\pi_{t+1}} \right)^{\frac{\mu_H}{1-\mu_H}} \Omega_{H,t+1}$$

and

$$\Upsilon_{H,t} = \lambda_t p_{H,t}^{\frac{\mu_H}{\mu_H - 1}} y_{H,t} + \beta\theta_H E_t \left(\frac{\pi_{t+1}^{\zeta}}{\pi_{t+1}} \right)^{\frac{1}{1-\mu_H}} \Upsilon_{H,t+1}$$

2.3.1 Exporters

Analogously to domestic firms, exporters behavior can be described as follows:

$$\tilde{p}_{H,t}^{*} = \mu_H^{*} \frac{\Omega_{H,t}^{*}}{\Upsilon_{H,t}^{*}}$$

where:

$$\Omega_{H,t}^{*} = \lambda_t p_{m,t} p_{H,t}^{*\frac{\mu_H^*}{\mu_H^* - 1}} y_{H,t}^{*} + \beta\theta_H^{*} E_t \left(\frac{\pi_{t+1}^{\zeta^*}}{\pi_{t+1}^{*}} \right)^{\frac{\mu_H^*}{1-\mu_H^*}} \Omega_{H,t+1}^{*}$$

and

$$\Upsilon_{H,t}^{*} = \lambda_t q_t p_{H,t}^{*\frac{\mu_H^*}{\mu_H^* - 1}} y_{H,t}^{*} + \beta\theta_H^{*} E_t \left(\frac{\pi_{t+1}^{\zeta^*}}{\pi_{t+1}^{*}} \right)^{\frac{1}{1-\mu_H^*}} \Upsilon_{H,t+1}^{*}$$

2.4 Foreign capital and intermediate goods firms

Similarly, for foreign firms we obtain:

$$y_{p,t}^{*} = k_t^{*} (A_t^{*} n_t^{*})^{1-\alpha*} - \phi^{*}$$

$$\frac{R_t^{k*}}{P_t^{*} w_t^{*}} = \frac{\alpha^{*}}{1-\alpha^{*}} \frac{n_t^{*}}{k_t^{*}}$$

$$p_{m,t}^{*} = \frac{1}{\alpha^{*}} \frac{R_t^{k*}}{P_t^{*}} (\frac{1-\alpha^{*}}{\alpha^{*}} A_t^{*} \frac{R_t^{k*}}{P_t^{*} w_t^{*}})^{\alpha^{*}-1}$$

$$y_{F,t+s}(i) = \left(\frac{\tilde{P}_{F,t}\,(i)\,\pi_{t+s}^{\zeta_F}}{P_{F,t+s}} \right)^{\frac{\mu_F}{1-\mu_F}} y_{F,t+s}$$

where $\pi_{t,t+s}^{\zeta_F} = \pi_{t+1}^{\zeta_F} \cdot \ldots \cdot \pi_{t+s}^{\zeta_F}$.

$$\tilde{p}_{F,t} = \mu_F \frac{\Omega_{F,t}}{\Upsilon_{F,t}}$$

where:

$$\Omega_{F,t} = \lambda_t^* p_{m,t}^* p_{F,t}^{\frac{\mu_F}{\mu_F - 1}} y_{F,t} + \beta^* \theta_F E_t \left(\frac{\pi_{t+1}^{\zeta_F}}{\pi_{t+1}} \right)^{\frac{\mu_F}{1-\mu_F}} \Omega_{F,t+1}$$

and

$$\Upsilon_{F,t} = \lambda_t^* q_t^{-1} p_{F,t}^{\frac{\mu_F}{\mu_F - 1}} y_{F,t} + \beta^* \theta_F E_t \left(\frac{\pi_{t+1}^{\zeta_F}}{\pi_{t+1}} \right)^{\frac{1}{1-\mu_F}} \Upsilon_{F,t+1}$$

$$\tilde{p}_{F,t}^* = \mu_F^* \frac{\Omega_{F,t}^*}{\Upsilon_{F,t}^*}$$

where:

$$\Omega_{F,t}^* = \lambda_t^* p_{m,t}^* p_{F,t}^{*\frac{\mu_F^*}{\mu_F^* - 1}} y_{F,t}^* + \beta^* \theta_F^* E_t \left(\frac{\pi_{t+1}^{\zeta_F^*}}{\pi_{t+1}^*} \right)^{\frac{\mu_F^*}{1-\mu_F^*}} \Omega_{F,t+1}^*$$

and

$$\Upsilon_{F,t}^* = \lambda_t^* p_{F,t}^{*\frac{\mu_F^*}{\mu_F^* - 1}} y_{F,t}^* + \beta^* \theta_F^* E_t \left(\frac{\pi_{t+1}^{\zeta_F^*}}{\pi_{t+1}^*} \right)^{\frac{1}{1-\mu_F^*}} \Upsilon_{F,t+1}^*$$

2.5 Aggregators

2.5.1 Home economy - domestic goods

Aggregators buy the intermediate goods from retailers. We assume perfect competition in this stage of the production process. They maximize profits, given by:

$$P_{H,t} y_{H,t} - \int P_{H,t}(i) y_{H,t}(i) di$$

subject to the technological constraint

$$y_{H,t} = \left(\int y_{H,t}(i)^{\frac{1}{\mu_H}} di \right)^{\mu_H}$$

From this problem we get the following first order conditions:

$$y_{H,t}(i) = \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{\frac{\mu_H}{1-\mu_H}} y_{H,t}$$

and the zero profit condition

$$P_{H,t} y_{H,t} - \int P_{H,t}(i) \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{\frac{\mu_H}{1-\mu_H}} y_{H,t} di = 0$$

$$P_{H,t}^{\frac{1}{1-\mu_H}} = \int P_{H,t}(i)^{\frac{1}{1-\mu_H}} di$$

The law of motion for the price index:

$$P_{H,t}^{\frac{1}{1-\mu_H}} = \theta_H \left(P_{H,t-1}\pi_t^{\zeta}\right)^{\frac{1}{1-\mu_H}} + (1-\theta_H)\left(\tilde{P}_{H,t}\right)^{\frac{1}{1-\mu_H}}$$

Dividing by $P_t^{\frac{1}{1-\mu_H}}$

$$p_{H,t}^{\frac{1}{1-\mu_H}} = \theta_H \left(p_{H,t-1}\frac{\pi_t^{\zeta}}{\pi_t}\right)^{\frac{1}{1-\mu_H}} + (1-\theta_H)\left(\tilde{p}_{H,t}\right)^{\frac{1}{1-\mu_H}}$$

2.5.2 Home economy - imported goods

Similarly, imported goods aggregators maximize profits given by:

$$P_{F,t}y_{F,t} - \int P_{F,t}(i)y_{F,t}(i)di$$

subject to the technological constraint

$$y_{F,t} = \left(\int y_{F,t}(i)^{\frac{1}{\mu_F}} di\right)^{\mu_F}$$

From this problem we get the following first order conditions:

$$y_{F,t}(i) = \left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{\frac{\mu_F}{1-\mu_F}} y_{F,t}$$

and the zero profit condition

$$P_{F,t}y_{F,t} - \int P_{F,t}(i)\left(\frac{P_{F,t}(i)}{P_{F,t}}\right)^{\frac{\mu_F}{1-\mu_F}} y_{F,t}di = 0$$

$$P_{F,t}^{\frac{1}{1-\mu_F}} = \int P_{F,t}(i)^{\frac{1}{1-\mu_F}} di$$

The law of motion for the price index:

$$P_{F,t}^{\frac{1}{1-\mu_F}} = \theta_F \left(P_{F,t-1}\pi_t^{\zeta_F}\right)^{\frac{1}{1-\mu_F}} + (1-\theta_F)\left(\tilde{P}_{F,t}\right)^{\frac{1}{1-\mu_F}}$$

Dividing by $P_t^{\frac{1}{1-\mu_F}}$

$$p_{F,t}^{\frac{1}{1-\mu_F}} = \theta_F \left(p_{F,t-1}\frac{\pi_t^{\zeta_F}}{\pi_t}\right)^{\frac{1}{1-\mu_F}} + (1-\theta_F)\left(\tilde{p}_{F,t}\right)^{\frac{1}{1-\mu_F}}$$

2.5.3 Foreign economy

Similarly, for the foreign economy we get:

$$y^*_{H,t}(i) = \left(\frac{P^*_{H,t}(i)}{P^*_{H,t}}\right)^{\frac{\mu^*_H}{1-\mu^*_H}} y^*_{H,t}$$

$$p^{*\frac{1}{1-\mu_{H^*}}}_{H,t} = \theta^*_H \left(p^*_{H,t-1}\frac{\pi^{*\zeta}_t}{\pi^*_t}\right)^{\frac{1}{1-\mu^*_H}} + (1-\theta^*_H)\left(\tilde{p}^*_{H,t}\right)^{\frac{1}{1-\mu^*_H}}$$

$$y^*_{F,t}(i) = \left(\frac{P^*_{F,t}(i)}{P^*_{F,t}}\right)^{\frac{\mu_{F*}}{1-\mu_{F*}}} y^*_{F,t}$$

$$p^{*\frac{1}{1-\mu_{F*}}}_{F,t} = \theta_{F^*} \left(p^*_{F,t-1}\frac{\pi^{\zeta^*_F}_t}{\pi^*_t}\right)^{\frac{1}{1-\mu_{F*}}} + (1-\theta_{F^*})\left(\tilde{p}^*_{F,t}\right)^{\frac{1}{1-\mu_{F*}}}$$

3 Goods market clearing

Below, we present equilibrium conditions that are neccessary for market clearing.

3.1 Volumes

For an individual firm i we have

$$y_{H,t}(i) + y^*_{H,t}(i) = k_t(i)^\alpha (z_t n_t(i))^{1-\alpha} - \phi$$

$$\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{\frac{\mu_H}{1-\mu_H}} y_{H,t} + \left(\frac{P^*_{H,t}(i)}{P^*_{H,t}}\right)^{\frac{\mu^*_H}{1-\mu^*_H}} y^*_{H,t} = k_t(i)^\alpha (z_t n_t(i))^{1-\alpha} - \phi$$

integrating over firms and taking into account country size gives

$$y_{H,t}\Delta_{H,t} + \frac{1-\omega}{\omega} y^*_{H,t}\Delta^*_{H,t} = k_t{}^\alpha (z_t n_t)^{1-\alpha} - \phi$$

where

$$\Delta_{H,t} = \int_0^1 \left(\frac{p_{H,t}(i)}{p_{H,t}}\right)^{\frac{\mu_H}{1-\mu_H}} di = \left(\frac{p_{H,t}}{p_{H,t-1}}\right)^{\frac{\mu_H}{\mu_H-1}} \theta_H \Delta_{H,t-1}\left(\frac{\pi_{\zeta_H,t}}{\pi_t}\right)^{\frac{-\mu_H}{\mu_H-1}} + (1-\theta_H)\left(\frac{\tilde{p}_{H,t}}{p_{H,t}}\right)^{\frac{-\mu_H}{\mu_H-1}}$$

And for the foreign economy:

$$\frac{\omega}{1-\omega} y_{F,t}\Delta_{F,t} + y^*_{F,t}\Delta^*_{F,t} \quad = k^*_t (z^*_t n^*_t)^{1-\alpha*} - \phi^*$$

$$\Delta_{F,t} = \left(\frac{p_{F,t}}{p_{F,t-1}}\right)^{\frac{-\mu_F}{\mu_F-1}} \theta_F \Delta_{F,t-1} \left(\frac{\pi_{\zeta_F,t}}{\pi^*_t}\right)^{\frac{\mu_F}{\mu_F-1}} + (1-\theta_F)\left(\frac{\tilde{p}_{F,t}}{p_{F,t}}\right)^{\frac{\mu_F}{\mu_F-1}}$$

$$\Delta^*_{F,t} = \left(\frac{p^*_{F,t}}{p^*_{F,t-1}}\right)^{\frac{-\mu^*_F}{\mu^*_F-1}} \theta^*_F \Delta^*_{F,t-1} \left(\frac{\pi^*_{\zeta_F,t}}{\pi^*_t}\right)^{\frac{\mu^*_F}{\mu^*_F-1}} + (1-\theta^*_F)\left(\frac{\tilde{p}^*_{F,t}}{p^*_{F,t}}\right)^{\frac{\mu^*_F}{\mu^*_F-1}}$$

3.2 Profits

Profits earned by individual firm i are (in real terms)

$$d_t(i) = p_{H,t}(i) y_{H,t}(i) + q_t p^*_{H,t}(i) y^*_{H,t}(i) - \frac{R^k_t}{P_t} k_t(i) + w_t n_t(i)$$

Integrating over firms and taking into account country size

$$d_t = p_{H,t} y_{H,t} + \frac{1-\omega}{\omega} q_t p^*_{H,t} y^*_{H,t} - w_t n_t - \frac{R^k_t}{P_t} k_t$$

Similarly:

$$d^*_t = \frac{\omega}{1-\omega} p_{F,t} y_{F,t} \frac{1}{q_t} + p^*_{F,t} y^*_{F,t} - w^*_t n^*_t - \frac{R^{k*}_t}{P^*_t} k^*_t$$

4 Bond market clearing and public sector:

Bond markets clear:

$$\omega B_{H,t} + (1-\omega) B^*_{H,t} = \omega B^g_{H,t}$$

$$\omega B_{F,t} + (1-\omega) B^*_{F,t} = (1-\omega) B^{g*}_{F,t}$$

Government budget constraint is given by:

$$B^g_{H,t} + T_t = R_{t-1} B^g_{H,t-1} + P_t g_t$$

Monetary policy is given by a standard Taylor rule:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\gamma_r} \left(\left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{gdp_t}{gdp}\right)^{\gamma_y}\right)^{1-\gamma_r} \exp\{\varepsilon_t^r\}$$

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6 Filtering problem

This section presents Kalman filtering problem faced by agents in both economies. Since productivity components (temporary and permanent) are not observed directly, agents infer them from the aggregated productivity and the signal about permanent technology growth.

Prediction equations There are six states in the filtering problem (all variables in logs): x_t, x_{t-1}, z_t, x_t^*, x_{t-1}^*, z_t^*. They follow:

$$x_t = (1+\rho^x)x_{t-1} - \rho^x x_{t-2} + \varepsilon_{X,t}$$

$$x_{t-1} = x_{t-1}$$

$$z_t = \rho^z z_{t-1} + \varepsilon_{Z,t}$$

$$x_t^* = (1+\rho^{x*})x_{t-1}^* - \rho^{x*} x_{t-2}^* + \varepsilon_{X,t}^*$$

$$x_{t-1}^* = x_{t-1}^*$$

$$z_t^* = \rho^{z*} z_{t-1}^* + \varepsilon_{Z,t}^*$$

where $\varepsilon_{X,t}$ and $\varepsilon_{X,t}^*$ are interpreted as permanent technology shocks while $\varepsilon_{Z,t}$ and $\varepsilon_{Z,t}^*$ are temporary technology shocks.

In the matrix form:

$$X_t = AX_{t-1} + BV_t^\epsilon$$

Thus, the prediction step in the Kalman filter is given by:

$$X_{t|t-1} = AX_{t-1|t-1}$$

$$P_{t|t-1} = AP_{t-1|t-1}A' + S_1$$

where $X_{t|t-1}$ denotes expected value of X in period t given informaiton set from period t-1, P - the covariance matrix,

$$A = \begin{bmatrix} 1+\rho^x & -\rho^x & 0 & & & \\ 1 & 0 & 0 & & 0 & \\ 0 & 0 & \rho^z & & & \\ & & & 1+\rho^{x*} & -\rho^{x*} & 0 \\ & 0 & & 1 & 0 & 0 \\ & & & 0 & 0 & \rho^{z*} \end{bmatrix}$$

S_1 depicts uncertainty around the prediction and is given by:

$$S_1 = \begin{bmatrix} (1-\rho^x)\sigma_x & 0 & 0 & & & \\ 0 & 0 & 0 & & 0 & \\ 0 & 0 & \sqrt{\rho^z}\sigma_z & & & \\ & & & (1-\rho^{x*})\sigma_x^* & 0 & 0 \\ & 0 & & 0 & 0 & 0 \\ & & & 0 & 0 & \sqrt{\rho^{z*}}\sigma_z^* \end{bmatrix} \bullet$$

$$\begin{bmatrix} (1-\rho^x)\sigma_x & 0 & 0 & & & \\ 0 & 0 & 0 & & 0 & \\ 0 & 0 & \sqrt{\rho^z}\sigma_z & & & \\ & & & (1-\rho^{x*})\sigma_x^* & 0 & 0 \\ & 0 & & 0 & 0 & 0 \\ & & & 0 & 0 & \sqrt{\rho^{z*}}\sigma_z^* \end{bmatrix} =$$

$$= \begin{bmatrix} (1-\rho^x)^2\sigma_x^2 & 0 & 0 & & & \\ 0 & 0 & 0 & & 0 & \\ 0 & 0 & \rho^z\sigma_z^2 & & & \\ & & & (1-\rho^{x*})^2(\sigma_x^*)^2 & 0 & 0 \\ & 0 & & 0 & 0 & 0 \\ & & & 0 & 0 & \rho^{z*}(\sigma_z^*)^2 \end{bmatrix}$$

Measurment equations and the Kalman gain System of measurment equations is:

$$a_t = (1-\lambda^x)x_t + \lambda^x x_t^* + z_t$$

$$a_t^* = x_t^* + z_t^*$$

$$s_t = x_t + \varepsilon_{s,t}$$

$$s_t^* = x_t^* + \varepsilon_{s,t}^*$$

In the matrix form:

$$S_t = CX_t + DV_t$$

where

$$C = \begin{bmatrix} 1-\lambda^x & 0 & 0 & \lambda^x & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_s & 0 \\ 0 & 0 & 0 & \sigma_s^* \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_s & 0 \\ 0 & 0 & 0 & \sigma_s^* \end{bmatrix} = $$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_s^2 & 0 \\ 0 & 0 & 0 & (\sigma_s^*)^2 \end{bmatrix}$$

The updated covariance matrix is then:

$$P_{t|t} = AP_{t-1|t-1}A' - KCAP_{t-1|t-1}A' + S_1$$

Note that $P_{t|t} = P_{t-1|t-1} = P$ as technology is a white noise process with constant variance. Matrix P is found as the fixed point of the system of equations:

$$K = PC'(CPC' + S_2)^{(-1)}$$

$$P = APA' - APC'K'A' + S_1$$

where the first equation is a standard formula for the Kalman gain.

6.1 Equivalence with the full-information model

From the Kalman filter we obtain :

$$X_{t|t} = AX_{t-1|t-1} + K(S_t - CAX_{t-1|t-1})$$

Using the predicted value for state variables, the measurment equation can be transformed:

$$S_t = CX_{t|t-1} + DV_t$$

$$S_t = CAX_{t-1|t-1} + DV_t$$

Thus:

$$X_{t|t} = AX_{t-1|t-1} + KDV_t$$

We want to obtain $\hat{V}_t = [e_1\, e_2]$ - vector of iid standard normal shocks (similarly to what we had earlier when we computed matrix S_1). Therefore, we express DV_t as:

$$DV_t = G\hat{V}_t$$

where matrix G is obtained from the factorization of covariance matrix

$$GG' = CPC' + S_2$$

Finally:

$$X_{t|t} = AX_{t-1|t-1} + KG\hat{V}_t$$

and

$$S_t = CAX_{t-1|t-1} + G\hat{V}_t$$

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