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FUZZY MULTIDIMENSIONAL POVERTY INDEX

ABSTRACT

Multidimensional Poverty Index (MPI) is a novel measure of poverty that is being used by United Nations Development Programme. The MPI is an answer to the call for a multidimensional view of poverty (Sen and Anand, 1997). It evaluates poverty at the individual level, with poor people being those who are deprived in multiple dimensions of well-being, and also the extent of poverty by assessing the intensity of deprivation. Formally, the MPI is a product of the headcount ratio and the proportion of dimensions in which individuals are deprived. The MPI methodology makes use of dual cutoff, that is, *within dimension cutoff* is the poverty line and *crossdimensional cutoff* is the minimum number of deprivations required in order to be considered poor. The arbitrariness of dual cutoff approach makes it natural to search for a fuzzy counterpart to the MPI. Fuzzy approach to poverty measurement depicts the vagueness that appears in the measurement of social phenomena instead of establishing single line that separates the poor and the non-poor. Majority of variables used in social measurement are of qualitative ordinal nature; any sharp boundary as to where, for example, illiteracy or malnutrition begins is discretionary. Instead, in fuzzy approach membership in poverty has various degrees, from low to severe. In this paper, we propose fuzzy counterpart to the MPI and discuss its properties and further generalizations.

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1. INTRODUCTION

It has been widely acknowledged for several decades that well-being and hence, the shortfall of well-being, i.e. poverty (deprivation), is a multidimensional concept. This idea has been expressed by many researchers with a most profound critique of the narrow focus of traditional measures on income presented by Nobel Prize laureate Amartya Sen, and further backed by such renowned initiatives as Stiglitz-Sen-Fitousi commission. The commission was created in 2008 on French government's initiative in order to better tackle the problem of measuring economic performance and social progress, in particular with respect to extending it beyond income and GDP. It is now commonly accepted that income poor do not necessarily have to be deprived in, for example, education or health, as in some countries public services are available. Indeed, as Alkire and Santos (2010) point out, there is nearly 40 percent of income poor in Ethiopia and almost 90 percent poor when health and education are also taken into account, and, on the other hand, almost 30 percent income poor in Uzbekistan and less than 5 percent poor with health and education considered. Income is not the sole indicator of poverty.

The United Nations have long underlined that a multidimensional approach to viewing poverty is necessary in order for the Millennium Development Goals to be achieved. As stated in June 2010 UNDP international assessment, there are significant interconnections among various dimensions of poverty deprivation. One important synergy concerns women empowerment and children health; in households where women are illiterate, child mortality is higher. Furthermore, removal of one barrier alleviates the impact of others. Many multidimensional measures of poverty have been constructed e.g. Tsui (2002), Bourguignon and Chakravarty (2003), Maasoumi and Lugo (2008). A measure which has recently gained in importance is the Multidimensional Poverty Index (MPI) developed by Oxford Poverty and Human Development Initiative (OPHI). Its advantage over the other measures is that it can be used with ordinal data (many well-being dimensions are ordinal), it provides for a more flexible identification strategy (typically multidimensional measures use union approach, namely, an individual is multidimensionally poor if he or she is poor in each dimension), and it can be decomposed into population groups. Thus we focus on this measure in our paper.

Alkire and Santos (2010) use MPI to evaluate poverty at the individual level for 104 countries. The measure has three dimensions: health, education and standard of living, which are weighted equally (1/3) and measured by 10 indicators with equal weights. These indicators are the following (Alkire and Santos 2010):

Education
- years of schooling (deprived if no household member has completed five years of schooling, weight 1/6)
- child enrollment (deprived if any school-aged child is not attending school in years 1 to 8, weight 1/6)

Health
- child mortality (deprived if any child has died, weight 1/6)
- nutrition (deprived if any adult or child is malnourished, weight 1/6)

Standard of Living
- electricity (deprived if the household has no electricity, weight 1/18)
- drinking water (deprived if the household does not have access to clean drinking water or clean water is more than 30 minutes walk from home, weight 1/18)
- sanitation (deprived if they do not have an improved toilet or if their toilet is shared, weight 1/18)
- flooring (deprived if the household has dirt, sand or dung floor, weight 1/18)
- cooking fuel (deprived if they cook with wood, charcoal or dung, weight 1/18)
- assets (deprived if the household does not own more than one of: radio, TV, telephone, bike, or motorbike, and do not own a car or tractor, weight 1/18)

A household is identified as poor if it is deprived in some combination of indicators whose weighted sum exceeds 30 percent of all deprivations. We leave detailed methodology on how the MPI is computed in section 2. Here, we only indicate dual cutoff that is used in calculating the index. First cutoff refers to poverty line, that is, for example in case of years of schooling it equals 5. In other words, only individuals with 5 or less years of schooling are classified as poor. Second cutoff is the 30 percent requirement in order for the person to be considered multidimensionally poor. This is how the MPI takes into account the extent of poverty deprivation. A person who is deprived in years of schooling and sanitation is not considered poor in this ranking because the sum of weights is less than 30 percent, that is, $1/6 + 1/18 < 3/10$. In order to be MPI poor one has to be deprived in at least two health/education indicators or all six standard of living indicators or 1 health/education and three standard of living indicators.

The implication of dual cutoff approach is arbitrariness involved in measurement. Let us consider the first cutoff and for example the dimension of *being educated*. Completing a few years of schooling is a poor indicator as it informs only about resources a person has received and not real competencies he or she acquired. Following Chiappero Martinetti (2006), being educated is an example of the functioning that is achieved gradually, ranging from illiteracy, through ability to write and read only but no formal education, the lowest levels of formal background but with cases of dropout etc., up to sufficient, good and very good levels of attendance in the schooling system and acquired abilities. Five years of formal education means absolutely different levels of relative education achievement in a developed and developing country. Also, in case of malnutrition, a low level of nutrition will differ depending on the person, if it is a child, a pregnant woman or a physical worker. As to the second cutoff, the MPI is an intermediate approach between union definition of poverty (poor if deprived in at least one dimension) and intersection definition (poor if deprived in all dimensions, here all 10 indicators). However, still with

30 percent cutoff line, a person who lives in destitute conditions, with no electricity, drinking water, cooking fuel and the appropriate floor is not considered MPI poor.

This dual arbitrariness involved in the MPI can be reconciled by adhering to fuzzy measures. Fuzzy measures distinguish between degrees in deprivation, therefore they treat differently expressions such as low, sufficient, quite good etc. These adjectives semantically represent different levels of intensity of phenomena and fuzzy approach aims to reflect this intensity. Therefore, with fuzzy measure individuals are not poor and non-poor, but rather poor to a certain degree. Of course, it does not matter for poverty whether someone is one or two percent poor, but it does whether a person is 70 or 90 percent poor. In this way, individual extent of poverty is explicitly taken into account by assigning to every person her degree of membership in poverty. The same applies to the number of dimensions of deprivation. The lowest required number is one dimension (union poverty) and the highest is all dimensions (intersection poverty). These can be viewed as extremes, with the former representing complete substitutability among dimensions (only one dimension is needed and it can be any) and the latter having zero level of substitutability among dimensions (all dimensions are necessary for achieving the state of being poor). If on the other hand we take weighted average of all dimensions, dimensions' degrees of substitutability will differ. There are many fuzzy aggregation functions which reflect intermediate levels of substitutability between dimensions.[1]

In this paper we propose fuzzy counterpart to the MPI, discuss its properties and offer further extensions. This article is organized as follows. In section 1 we show in detail how the MPI is computed. In section 2 we present main concepts in fuzzy approach to poverty measurement. In section 3 we introduce fuzzy MPI In section 4 we discuss poverty measurement axioms that fuzzy MPI fulfills. In section 5 we formulate a generalization of fuzzy MPI. Finally we conclude.

2. THE MPI METHODOLOGY: ALKIRE FOSTER METHOD

The MPI is a member of a family of poverty measures proposed by Alkire and Foster (2007, 2009). It is called Adjusted Headcount as it is headcount index (percentage of poor in the society) multiplied by average number of deprivations. We will now present Alkire-Foster methodology.[2]

Let $\mathbb{R}_{\geq 0}$ the space of positive reals with zero. Further, let $\mathbb{M}^{n \times d}$ denote the set of all $n \times d$ real-valued matrices and $X \in M^{n \times d}$. The element $x_{ij} \in X$ denotes the value of attribute j for the i-th person; $x_{.j} \in \mathbb{R}^n$, $x_{.j} = (x_{ij})_{i=1}^{n}$ and

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[1] We leave more detailed discussion to section 2. Please see also Zimmermann (1990).

[2] Unlike Alkire and Foster (2007, 2009), who first consider dimensions with equal weights, we already start with dimensions that potentially are weighted differently.

$x_{i.} \in \mathbb{R}^d$, $x_{i.} = (x_{ij})_{j=1}^d$ denote, respectively, the column and the row of matrix X and, in the same order, the quantity of attribute j possessed by every person and the values of every attribute for person i-th. In what follows we consider the case of a population of a given size n.[3] Let M: $\mathbb{M}^{n \times d} \mapsto \mathbb{R}_{\geq 0}$ denote the poverty index. Let $w \in \mathbb{R}_{\geq 0}^d$ be the vector of weights and $\sum_{j=1}^d w_j = d$.[4] Let $z = (z_j)_{j=1}^d \in \mathbb{R}^d$ be the vector of poverty lines for every dimension.

We start by assuming that all weights are equal to one. We define the normalized gap for each person as

$$g_{ij} = \begin{cases} \dfrac{z_j - x_{ij}}{z_j}, & x_{ij} < z_j \\ 0, & x_{ij} \geq z_j. \end{cases} \tag{1}$$

That is, we assign zero if an individual value is above the poverty threshold for the j-th dimension z_j. Now define matrix $g_\alpha(X) \in M^{n \times d}$ with the entries g_{ij}^α, that is, (1) raised to the power of α. Now we construct a vector of deprivation $c = (c_i)_{i=1}^n \in \mathbb{R}^n$, whose i-th entry $c_i\left(x_{i.}\right) = \sum_{j=1}^d g_0\left(x_{ij}\right)$ gives the sum of dimensions in which i is deprived. Let ρ: $\mathbb{R}_{\geq 0}^d \times \mathbb{R}_{\geq 0}^d \mapsto \{0,1\}$ be the identification function which identifies person as poor if $c_i \geq k$ (in which case $\rho(x_{i.}, z) = 1$) and non-poor if $c_i < k$ (in which case $\rho(x_{i.}, z) = 0$). Thus an individual is considered poor if his deprivation count is greater than or equal to k. With equal weighting, where each dimension gets weight one, as is the case now, we say that a multidimensionally poor person is the one who is deprived in at least k dimensions.

The normalized gaps of individuals who are not identified as multidimensionally poor are replaced by zero. Name this censored normalized gaps as

$$\bar{g}_{ij} = \begin{cases} g_{ij}, & c_i \geq k \\ 0, & c_i < k \end{cases}. \tag{2}$$

We now define a matrix $\bar{g}_\alpha(k)$ with entries $\bar{g}_{ij}^\alpha$, that is (2) raised to the power α. The multidimensional Alkire-Foster poverty measure is obtained as the mean of this matrix

$$M_\alpha\left(X,z\right) = \mu\left(\bar{g}_\alpha(k)\right), \qquad \alpha \geq 0. \tag{3}$$

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[3] However, the index turns out to be measuring poverty relative to population sizes so it can be used for comparisons of two populations of different sizes – please see section 4: Replication Invariance axiom.

[4] Please note that weights of dimensions presented in the previous section sum to one, not 10. This is for the ease of exposition. Weights normalized to 10 would be $\dfrac{10}{6}$ (instead of $\dfrac{1}{6}$) and $\dfrac{10}{18}$ (instead of $\dfrac{1}{18}$).

When $\alpha = 0$ we obtain Adjusted Headcount or M_0 which is the mean of the average of deprivation counts. When $\alpha = 1$ the measure is called Adjusted Poverty Gap or M_1. The presented approach is called dual cutoff since it uses two cutoffs to determine who is poor: within dimension cutoff z and cross-dimensional cutoff k.

Example 1
Let us consider the following matrix X of four individuals and three columns:

$$
\begin{pmatrix}
10 & 4 & 5 \\
5 & 1 & 1 \\
1 & 1 & 2 \\
4 & 3 & 10
\end{pmatrix}
$$

Let vector of weights be $w = (1, 1, 1)$ and poverty line be $z = (3, 2, 6)$ and $k = 2$. The $g_0(X)$ matrix is

$$
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

and $c = (1, 2, 3, 0)$. Since $k = 2$ a second and third person is identified as poor. Thus matrix $\bar{g}_0(2)$ is

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

Censored vector of weighted deprivations is $\bar{c} = (0, 2, 3, 0)$ and $M_0 = \dfrac{5}{12}$. The MPI is M_0 with dimensions described in the Introduction and the following vector of weights $w = \left(\dfrac{10}{6}, \dfrac{10}{6}, \dfrac{10}{6}, \dfrac{10}{6}, \dfrac{10}{18}, \dfrac{10}{18}, \dfrac{10}{18}, \dfrac{10}{18}, \dfrac{10}{18}, \dfrac{10}{18} \right)$, where first weight is for the years of schooling and last for assets.

Now let $\alpha = 2$ and the $g_2(X)$ is

$$
\begin{pmatrix}
0 & 0 & \left(\dfrac{1}{6}\right)^2 \\
0 & \left(\dfrac{1}{2}\right)^2 & \left(\dfrac{5}{6}\right)^2 \\
\left(\dfrac{2}{3}\right)^2 & \left(\dfrac{1}{2}\right)^2 & \left(\dfrac{3}{6}\right)^2 \\
0 & 0 & 0
\end{pmatrix}.
$$

Again, second and third person are identified as multidimensionally poor, hence $\bar{g}_2(2)$ is

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & \left(\dfrac{1}{2}\right)^2 & \left(\dfrac{5}{6}\right)^2 \\
\left(\dfrac{2}{3}\right)^2 & \left(\dfrac{1}{2}\right)^2 & \left(\dfrac{3}{6}\right)^2 \\
0 & 0 & 0
\end{pmatrix}
$$

and $M_2 = \dfrac{1}{12}\left(\dfrac{1}{2}^2 + \dfrac{5}{6}^2 + \dfrac{2}{3}^2 + \dfrac{1}{2}^2 + \dfrac{3}{6}^2\right)$.

For the weighted case the deprivation count vector is not a sum of deprivations but a sum of weighted deprivations with entries $c_i\left(x_{i.},w\right) = \sum_{j=1}^{d} w_j g_\alpha\left(x_{ij}\right)$ for each person. The dimensional cutoff k now ranges between $k = \min\{w_j\}$ and $k = d$. Measures are still obtained as means of appropriate matrices.

Example 2

Continuing previous example we now have a vector of weights $w = \left(2, \dfrac{1}{2}, \dfrac{1}{2}\right)$. Matrix $g_0(X)$ is as before

$$
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}.
$$

The vector of weighted deprivation counts is $c = \left(\dfrac{1}{2}, 1, 3, 0\right)$. Since $k = 2$ only third person is now identified as multidimensionally poor, so $M_0 = \dfrac{3}{12}$. Lower weight given to dimensions in which most individuals are deprived (second and third column) results in lower multidimensional poverty score.

3. FUZZY APPROACH TO POVERTY MEASUREMENT

Fuzzy set theory has been applied successfully to poverty measurement (Lemmi and Betti, 2006) and helps to represent gradual membership in poverty, both for quantitative and qualitative variables. As Chiappero-Marinetti (2006) points out "in fuzzy environment, the conventional hard threshold, which determines an unambiguous distinction between "poor" and "not poor", is substituted by a soft

threshold that depicts an intermediate, gradual representation between acceptable and unacceptable living conditions, or adequate and inadequate levels of well-being, without establishing a single abrupt cutoff line." This gradual membership is typically expressed by the so called fuzzy membership function. It replaces standard poverty line identification function which gives only values zero (non-poor) and one (poor). To every individual this function assigns his or her degree of membership in poverty, which is a number from the interval [0, 1]. We first give an example of the fuzzy membership function and then we define it.

Example 3

Consider matrix X in *Example 1* and the first dimension with values (10, 5, 1, 4) and its cutoff line $z_1 = 3$. Instead of only zero and one, let membership function μ_{ij} be the following

$$\mu_{ij}\left(x_{.j}\right) = \begin{cases} 0, & x_{ij} > 6 \\ 0.5, & x_{ij} \in [3, 6]. \\ 1, & x_{ij} < 3 \end{cases} \tag{4}$$

With this membership function, the first column of the matrix g_0 would be (0, 0.5, 1, 0.5) instead of (0, 0, 1, 0). With the latter, individuals first and fourth are both considered non-poor although their values differ significantly (10 and 4). New membership function reflects this fact by assigning membership level one-half to the fourth person and zero to the first.

Let us now consider again the same matrix and second column with values (4, 1, 1, 3) and cutoff $z_2 = 2$ and the following membership function $\mu_{ij}\left(x_{.j}\right) = \dfrac{\max\{x_{.j}\} - x_{ij}}{max\{x_{.j}\}}$, which equals 0 for a person with the highest value of attribute j. Then in the second column of the transformed matrix g_0 we would get values $\left(0, 1, 1, \dfrac{1}{3}\right)$. Here individual fourth is treated differently than individual first comparing to the standard case. This is to reflect that their values (3 and 4) vary.

A *fuzzy membership function* for attribute j is a vector function $\mu_{.j}: \mathbb{R}^n \mapsto [0,1]^n$, where $\mu_{ij}: \mathbb{R}^n \mapsto [0,1]$ denotes the membership function of i-th person in attribute j. We also denote $\mu_{i.} = (\mu_{ij})_{j=1}^d$ and $\mu_{i.}: M^{n \times d} \mapsto [0,1]^d$; that is, $\mu_{i.}$ gives for the i-th person the value of her membership in every dimension, or as we also call it, her membership in the whole *functioning*. Finally, $\mu = (\mu_{ij})_{i,j} = (\mu_{i.})_{i=1}^n$.[5] As to aggre-

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[5] It is important to note the definition of $\mu_{.j}$ and μ_{ij}. Chakravarty (2006) defines $\mu_{.j}$ as a function of attribute j which takes x_{ij}, that is, a value of attribute j for person i and gives back the number between [0,1] which should be interpreted as the degree of membership of person i in attribute j. This however excludes a priori reasonable membership functions such as the second function in *Example 3* (see also Chiappero Martinetti (2006), which depends on the whole vector $x_{.j}$.

gating dimensions, different *fuzzy aggregation operators*[6] are used. Most common fuzzy aggregation functions are $m_{\min}^f(p) = \min_l(p_l)$ and $m_{\max}^f(p) = \max_l(p_l)$, where $p = (p_l)_{l=1}^d$. These two aggregators correspond to, respectively, intersection and union definition of poverty.[7] Intersection definition assumes that a person is poor only if it is poor in each dimension. In contrast, it suffices to be poor in at least one dimension in order to be considered multidimensionally poor according to union approach. An example will clarify.

Example 4

Let us consider the fourth individual in *Example 1* and let fuzzy membership function be the second function in *Example 3*. Her degrees of membership in poverty are now $\left(\dfrac{2}{3}, \dfrac{1}{3}, 0\right)$. Then $m_{\min}^f = 0$ and $m_{\max}^f(p) = \dfrac{2}{3}$. As we said, in order to be intersection poor individual needs to have positive degrees of membership in every dimension, and since he does not have, he is not considered intersection poor, that is, $m_{\min}^f = 0$. In contrast, with union poverty individual fourth is assigned a significant degree of poverty deprivation, namely, $\dfrac{2}{3}$ although he is not deprived in the third dimension.

Fuzzy intersection and union aggregation operators are in a sense special cases of the general aggregation function (that is, min, max operators are its limits with respect to θ):

$$m^f\left(p\right) = \left(\frac{1}{d}\sum_{l=1}^d \left(p_l\right)^\theta\right)^{\left(\frac{1}{\theta}\right)}$$

The lower the θ, the more weight is given to small values in the aggregation. Intersection definition of poverty admits full compensation in terms of degree of membership in poverty (high degrees of membership are compensated by the lower degree and thus, the overall degree is low), whereas union definition allows zero compensation among dimensions with respect to poverty deprivation.[8] Therefore, taking into account that intersection and union aggregation operators are limits of the general aggregation function, generalized mean aggregation function can be viewed as admitting all intermediate levels of compensation among dimensions. Lower values of θ indicate higher degrees of compensation with regard to poverty deprivation membership.

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[6] The literature on fuzzy aggregation operators is very rich, here we briefly present only main concepts, for more detail please refer to Zimmermann (1990).

[7] For a survey on fuzzy aggregators used in poverty measurement see Chiappero Martinetti (2006)

[8] When we consider membership in well-being instead of poverty, as in Chiappero Marinetti (2006), that is, the higher score the better, then the reverse holds, namely intersection equals no compensation and union means full compensation among dimensions.

Further, let $M_\mu^f: M^{n \times d} \mapsto \mathbb{R}_{\geq 0}$ denote the fuzzy poverty index and $M_\mu^f(X)$ is the value of a fuzzy poverty index for matrix X. In the paper we consider only poverty indices that are subgroup decomposable, so that it can be written that

$$M_\mu^f(X) = \frac{1}{n}\sum_{i=1}^n m^f\left(\mu_{i.}(X)\right),$$

which means that overall poverty index is the sum of the individual fuzzy aggregation functions $m^f\left(\mu_{i.}(X)\right)$.

4. FUZZY MULTIDIMENSIONAL POVERTY INDICES

In order to define a fuzzy counterpart to the MPI and, in general, to the M_α family of indices presented in section 1, we shall pick appropriate fuzzy membership function and fuzzy aggregation operator. We claim that a natural candidate for a fuzzy membership function is normalized gap function which we introduced in section 1. Formal definition is as follows:

$$\bar{\mu}_{ij,\alpha,z} = \begin{cases} \left(\dfrac{z - x_{ij}}{z}\right)^\alpha, & x_{ij} < z \\ 0, & x_{ij} \geq z \end{cases}, \tag{5}$$

where z is a parameter (poverty line) and also α; $z \in \mathbb{R}$; $\alpha \in \mathbb{R}_{\geq 0}$.

Thus, indices with parameter $\alpha \neq 0$ can all be viewed as fuzzy extensions to the MPI with respect to evaluating poverty membership degree. By transforming each entry of the original matrix X by a normalized gap function raised to the power of α, we will obtain a transformed matrix g_α; in what follows, in order to ease notation we only put parameter α and skip g_α.

The real problem lies in choosing the appropriate aggregation function. In this section we propose one. First, we describe the idea behind it. In the MPI methodology, cross-dimensional cutoff gives a number of dimensions of deprivation that is necessary for an individual to be considered multidimensionally poor. In order for this barrier not to be so clear-cut we would like our fuzzy index to take into account to some extent individuals who are deprived in less than the number of dimensions imposed by the cutoff. Therefore, the fuzzy index will be a family of indices parametrized by two numbers: k is the cross-dimensional cutoff in the standard MPI index and $s \in [0,1]$ indicates to what extent we value those deprived in less than k dimensions.

Definition 1. *Fuzzy MPI Let $X \in M^{n \times d}$ and vector of weights be $w \in \mathbb{R}_{\geq 0}^k$. Let $g_\alpha(X)$ be defined as in Section 1. We define a family of functions $\bar{m}_k^f: [0,1]^n \to [0,1]$ by*

$$\bar{m}_k^f\left(g_\alpha(x_{i.})\right) = \begin{cases} \displaystyle\sum_{j=1}^d w_j g_\alpha(x_{ij}) & if \displaystyle\sum_{j=1}^d w_j g_0(x_{ij}) \geq k \\ 0 & if\ otherwise \end{cases}. \tag{6}$$

We have two parameters $k \in \mathbb{R}_{\geq 0}$, $k \leq d$ and $s \in [0,1]$. We define $\bar{\mathrm{m}}_{k,s}^f : [0,1]^n \to [0,1]$ by

$$\bar{m}_k^f\left(g_\alpha\left(x_{i.}\right)\right) = s\bar{m}_0^f\left(g_\alpha\left(x_{i.}\right)\right) + \left(1-s\right)\bar{m}_k^f\left(g_\alpha\left(x_{i.}\right)\right). \tag{7}$$

Fuzzy $M_{\alpha,k,s}^f$ index is defined as

$$M_{\alpha,k,s}^f\left(X\right) = \frac{1}{n}\sum_{i=1}^{n}\bar{m}_{k,s}^f \quad \alpha, k \in \mathbb{R}_{\geq 0}, k \leq d, s \in [0,1], \tag{8}$$

where $\bar{m}_{k,s}^f$ is defined by (7).

Remark 3.1. For $k = 0$ we have $\bar{m}_{0,s}^f = m_0^f = \dfrac{1}{d}\sum_{j=1}^{d} w_j g_\alpha\left(x_{ij}\right)$.

Remark 3.2. For $s = 1$ we have $\bar{m}_{k,1}^f = m_{0,1}^f = m_0^f$.

Remark 3.3. **Fuzzy MPI** is $M_{\alpha,k,s}^f$, k = 1, ..., 10, dimensions described in the Introduction and a vector of weights $w = \left(\dfrac{10}{6}, \dfrac{10}{6}, \dfrac{10}{6}, \dfrac{10}{6}, \dfrac{10}{18}, \dfrac{10}{18}, \dfrac{10}{18}, \dfrac{10}{18}, \dfrac{10}{18}, \dfrac{10}{18}\right)$.

Example 5

Let $g_1(X)$ be the following and $w = (1, 1, 1, 1)$:

$$\begin{pmatrix} 1 & 1 & 1 & \dfrac{1}{2} \\ \dfrac{2}{3} & \dfrac{1}{2} & \dfrac{1}{7} & 0 \\ \dfrac{1}{2} & \dfrac{1}{10} & 0 & 0 \\ \dfrac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

We will now calculate poverty levels for different members of the family of indices (4).

$$M_{\alpha=1,k=1,s}^f\left(X\right) = M_{\alpha=1,k=1}^f\left(X\right) = \frac{1}{4}\left(\frac{7}{4\times 2} + \frac{55}{4\times 42} + \frac{6}{4\times 10} + \frac{1}{4\times 2}\right) \approx 0.37$$

Taking into account Remark 3.1, this should be equal to the value of standard M_1 index, which is

$$M_1\left(\bar{X}\right) = \frac{1}{16}\left(1+1+1+\frac{1}{2}+\frac{2}{3}+\frac{1}{2}+\frac{1}{7}+\frac{1}{2}+\frac{1}{10}+\frac{1}{2}\right) \approx 0.37$$

Let us now consider $M_{\alpha=1,k=2,s=0.5}^f$ $M_{\alpha=0,k=4,s=0.2}^f$ and:

$$M_{\alpha=1,k=4,s=0.5}^f\left(X\right) = \frac{1}{4}\left(\frac{7}{4\times 2} + \frac{55}{4\times 42} + \frac{6}{4\times 10} + \frac{1}{4\times 2}\frac{1}{2}\right) \approx 0.35$$

$$M_{\alpha=1,k=4,s=0.2}^f\left(X\right) = \frac{1}{4}\left(\frac{7}{4\times 2} + \frac{2}{10}\frac{55}{4\times 42} + \frac{2}{10}\frac{6}{4\times 10} + \frac{2}{10}\frac{1}{4\times 2}\frac{1}{2}\right) \approx 0.25$$

5. PROPERTIES

In this section we will review fuzzy index (4) against poverty measurement properties presented in Chakravarty (2006) and Alkire and Foster (2007, 2009). Let us define set $S_j = \{i : g_\alpha(x_{ij}) > 0\}$, that is, S_j is the set of individuals who are deprived in the j-th dimension.

Poverty Focus. For all X, $X' \in M^{n \times d}$, if for all j: $S_{\mu,j}(X) = S_{\mu,j}(X')$ and $x_{ij} = x'_{ij}$ for all $i \in S_{\mu,j}(X)$, then $M^f_\mu(X) = M^f_\mu(X')$.

For fixed parameters k, s $M^f_{\alpha,k,s}$ fulfills Focus axiom. If between two matrices of data on each dimension the set of individuals who have positive values of poverty score does not change and values of their poverty scores do not change either, then a given fuzzy MPI will be the same for both matrices. In other words, individuals who are said to be rich, in a sense that they have zero poverty score on the considered dimension do not influence the poverty assessment. An example where this axiom is not satisfied is fuzzy membership function given in *Example 1*. As we can see, even if we do not change the values of attributes of the poor and the set of poor persons, for (1) to change it suffices for the value of the richest person to be higher under X than X'. Although a function such as the one in *Example 1* is considered a reasonable membership function (see Chiappero Martinetti (2006)), it is impossible for (1) to fulfill the basic axiom of poverty measurement using this membership function.

The above is the definition of Poverty Focus axiom by Chakravarty (2006). Let us notice that Chakravarty's Poverty Focus axiom corresponds to Deprivation Focus by Alkire and Foster (2007, 2009). In Alkire and Foster, Deprivation Focus requires that an increment to a nondeprived dimension of an individual, that is, to a dimension in which individual's value is above poverty line, should not affect poverty level (whether an individual is considered poor or not). The same is given above in a formal way. An increment to the non-deprived dimension does not change the set of individuals deprived in this dimension nor the values of deprived. Thus, it should not change poverty level. Poverty Focus axiom in Alkire and Foster concerns only the non-poor. If anyone outside the set of individuals who are identified as poor experiences an increment (to any dimension), then poverty index should not change. In contrast, notice that in the Deprivation Focus no identification method of the poor is needed; Deprivation axiom extends to both poor and non-poor. If identified as poor are individuals deprived in at least k dimensions, then clearly for $s \neq 0$ fuzzy index $M^f_{\alpha,k,s}$ changes when the values of non-poor change. On the other hand, for $s = 1$, as 3.3 suggests, we are back in the union definition of poverty: poor is someone who is deprived in any dimension. Then, Poverty Focus as understood by Alkire and Foster will be satisfied, but also, for union identification their Deprivation axiom implies Poverty axiom.[9]

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[9] See Alkire and Foster (2007) footnote 26.

Normalization *For all $X \in M^{n \times d}$ if for all j, $S_{\mu,j}(X)$ is an empty set, then $M_{\mu}^{f}(X) = 0$.*

Fuzzy MPI fulfills Normalization. If no individual has positive poverty score on every dimension then $M_{\alpha,k,s}^{f}$ will return zero. For indices with parameters $s \neq 0$ the reverse is true. For indices with $s = 0$ it may happen that, for example, for the following matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ \dfrac{1}{10} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we obtain $M_{\alpha,k=2,s=0}^{f} = 0$. For other indices $M_{\alpha,k,s}^{f}$ is zero if only if $S_{\mu,j}(X)$ is an empty set.

Monotonicity *For all X, X' $\in M^{n \times d}$ if $x_{rl} = x'_{rl}$ for all $r \in \dfrac{\{1, ..., n\}}{\{i\}}$ and $x_{il} = x'_{il}$ for all $l \in \dfrac{\{1, ..., d\}}{\{j\}}$ and $x_{ij} < x'_{ij}$, where $i \in S_{\mu,j}(X)$ then $M_{\mu}^{f}(X') < M_{\mu}^{f}(X)$.*

Again, except for indices with parameters $s \neq 0$ Monotonicity is fulfilled. For indices with parameters $s = 0$ it may happen that for the following matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ \dfrac{1}{10} & \dfrac{1}{10} & 0 \\ \dfrac{1}{3} & 0 & 0 \end{pmatrix}$$

and another matrix, in which the value of the last person decreases

$$\begin{pmatrix} 1 & 1 & 1 \\ \dfrac{1}{10} & \dfrac{1}{10} & 0 \\ \dfrac{1}{4} & 0 & 0 \end{pmatrix}$$

$M_{\alpha,k=2,s=0}^{f}$ stays the same since due to $s = 0$ this index gives zero weight to the last person so it will be insensitive to any changes of the last person situation (unless it becomes deprived in at least one more dimension, but this is not our focus here).

Transfers Principle *Let X' = BX, where B is a bistochastic not permutation matrix. Then $M_{\mu}^{f}(X') \leq M_{\mu}^{f}(X)$.*

A bistochastic matrix is a nonnegative matrix, in which the elements in row and columns sum to one. After multiplication by a bistochastic matrix, matrix is less spread out and so it matrix transformed by a normalized gap function. If a transfer is made from the poor person to the poorer, $M_{\alpha,k,s}^{f}$ fulfills Transfer axiom in the same manner as M_{α}^{f} indices do (proof in Alkire and Foster (2007

p. 17)). The same applies when the transfer is made from an individual who is not considered poor but the transfer is such that he stays rich afterwards. The problem appears when the transfer is such that following the transfer rich becomes poor in a given dimension. His poverty score then changes discontinuously and overall poverty may not go down, as it happens for the following matrix:

$$\begin{pmatrix} 0.8 & 0 \\ 1.1 & 0 \end{pmatrix}$$

Let poverty cut be $z = (1, 1)$. Hence, before the transfer (with equal weighing of dimensions) we have $M^f_{\alpha=1,k=2,s=0.25} = \frac{1}{4}\left(\frac{1-0.8}{1} + 1 + \frac{1}{4}\right) = \frac{3}{10} + \frac{1}{16} = 0.3625$.

Let us now introduce the transfer which transforms the original matrix into

$$\begin{pmatrix} 0.94 & 0 \\ 0.96 & 0 \end{pmatrix}$$

Now we have $M^f_{\alpha=1,k=2,s=0.25} = \frac{1}{4}\left(\frac{1-0.94}{1} + 1 + \frac{1-0.96}{1} + 1\right) = \frac{1}{4}\left(1.06 + 1.04\right) = 0.525$.

Symmetry *For all $X \in M^{n \times d}$, $M^f_\mu(X) = M^f_\mu(X')$, whenever X' is obtained from X by permuting individuals.*

Fuzzy MPI fulfills Symmetry.

Continuity *M^f_μ is continuous on $M^{n \times d}$.*

Fuzzy MPI in general is not continuous, but this is because it resembles standard MPI, which is discontinuous too. For example, let us consider $M^f_{\alpha,k=3,s=0}$ and a person who is deprived in two dimensions. Then an arbitrarily small increase in poverty score on some other dimension is associated with significant change in the overall poverty score (from zero to the mean of poverty scores in all dimensions) as this person becomes deprived in three dimensions.

Increasingness in membership function *For all $X \in M^{n \times d}$, if $S_{\mu.j}(X) = S_{\mu'.j}(X)$ and $\mu_h = \mu'_h$ for all $h \in \{1, ..., d\} - \{j\}$ and $\mu_{ij}(x_j) < \mu'_{ij}(x_j)$ for all $i \in S_{\mu.j}$, then $M^f_{\mu'}(X) < M^f_\mu(X)$.*

Fuzzy MPI fulfills Increasingness in membership function. If on a given dimension everyone gets a higher score then these higher values affect individual poverty scores (weighted means) and hence the value of overall poverty too.

Scale invariance *For all $X \in M^{n \times d}$, $M^f_\mu(\Omega X^T) = M^f_\mu(X)$, where $\Omega = diag(\omega_1, ..., \omega_k)$, $\omega_j > 0$ for all j and X^T is the transpose of a matrix X.*

Whether $M^f_{\alpha,k,s}$ fulfills Scale Invariance depends on the fuzzy membership function. Normalized gap function (2) is scale invariant if we look at it as a function of both x_{ij} and z, that is, when we scale up/down x_{ij}, poverty line changes its scale appropriately. Thus, $M^f_{\alpha,k,s}$ fulfills Scale Invariance.

Decomposability *Let $X \in M^{n \times d}$ and $X' \in M^{m \times d}$. We merge X, X' to obtain $(X, X') \in M^{(n+m) \times d}$. Index M_μ^f is decomposable if we can write $M_\mu^f\left(\left(X, X'\right)\right) =$*

$$= \frac{n}{n+m} M_\mu^f\left(X\right) + \frac{m}{n+m} M_\mu^f\left(X'\right).$$

Fuzzy MPI is decomposable as it is the average of $\bar{m}_{k,s}^f$, which are defined for every individual. It follows that fuzzy MPI is also replication invariant as next axiom requires.

Replication Invariance *Let $X' \in M^{mn \times d}$ be a replication of $X \in M^{n \times d}$, that is $X' = (X, X, ..., X)$, where X appears m times. Then $M_\mu^f(X') = M_\mu^f(X)$.*

In other words, fuzzy MPI measures poverty relative to the population sizes and can be used for making comparisons of the population of different sizes.

6. FURTHER EXTENSIONS

Fuzzy poverty indices (7) weigh by parameter s individuals who are deprived in less than k dimensions. Individuals who are deprived in $k-1, k-2, ..., 1$ dimensions are all treated the same. In this section our goal is to generalize $M_{\alpha,k,s}^f$ indices so that individuals deprived in more dimensions (but still less than k) are given greater weight. This can be seen as a similar generalization as with differing dimensions' weights. Here we differentiate weights given to different degrees of being multidimensionally poor. For example, it might be that policymakers want to target multidimensionally poor, but in particular they want to target those deprived in more dimensions. Instead of arbitrarily targeting only those deprived in all dimensions, one may want to include into their objective function different value judgments about specifically multidimensionality aspect of poverty. For example, a widely used definitions of poverty such as union approach (Alkire and Foster 2009) treats as poor only those deprived in any dimensions. This means that in the evaluation we offer, it is enough that the weight attached to deprivation in just one dimension is different from zero for a given individual to be counted as poor. On the other hand, in another widely used definition of poverty, namely the intersection approach, only individuals who are deprived in all dimensions are considered as poor. In the generalized fuzzy MPI we introduce below this means that the weights correspond to a maximum number of deprivations ($k = d$) and they are all equal and different than zero. In general, however, there can be intermediate definitions of poverty between union and intersection. The generalized fuzzy MPI allows for such flexibility.

Definition 2. Generalized fuzzy MPI *Let $X \in M^{n \times d}$ and vector of weights be $w \in \mathbb{R}_{\geq 0}^d$ and $s = \left(s_l\right)_{l=1}^{k-1}$, $s_l \in [0,1]$; $\sum_{l=1}^{k-1} s_l \leq 1$. Let $g_\alpha(X)$ be defined as in Section 1. We define a family of functions $\bar{m}_k^f : [0,1]^n \to [0,1]$ by*

$$\bar{m}_k^f\left(g_\alpha\left(x_{i\cdot}\right)\right) = \begin{cases} \sum_{j=1}^d w_j g_\alpha\left(x_{ij}\right) & if \sum_{j=1}^d w_j g_0\left(x_{ij}\right) \geq k \\ 0 & if \ otherwise \end{cases}. \tag{9}$$

We have parameters $k \in \mathbb{R}_{\geq 0}$, $k \leq d$ and $s \in [0,1]^{k-1}$. We define $\bar{m}_{k,s}^{gf} : [0,1]^k \to [0,1]$ by

$$\bar{m}_{k,s}^{gf}\left(g_\alpha\left(x_{i\cdot}\right)\right) = s_1 \bar{m}_1^f\left(g_\alpha\left(x_{i\cdot}\right)\right) + \dots + \left(1 - s_1 - \dots - s_{k-1}\right)\bar{m}_k^f\left(g_\alpha\left(x_{i\cdot}\right)\right). \tag{10}$$

Generalized fuzzy $M_{\alpha,k,s}^f$ index *is defined as*

$$M_{\alpha,k,s}^f\left(X\right) = \frac{1}{n}\sum_{i=1}^n \bar{m}_{k,s}^{gf} \qquad \alpha, k \in \mathbb{R}_+, k \leq d, s \in [0,1]^{k-1}, \tag{11}$$

where $\bar{m}_{k,s}^{gf}$ is defined by (10).

Remark 5.1. **Generalized fuzzy MPI** is $M_{\alpha,k,s}^f$, $k = 1, \dots, 10$, dimensions described in the Introduction and a vector of weights

$$w = \left(\frac{10}{6}, \frac{10}{6}, \frac{10}{6}, \frac{10}{6}, \frac{10}{18}, \frac{10}{18}, \frac{10}{18}, \frac{10}{18}, \frac{10}{18}, \frac{10}{18}\right).$$

CONCLUSION

In this article we proposed a fuzzy extension to the novel poverty measure which is Multidimensional Poverty Index. This index is a member of the family of indices proposed by Alkire and Foster (2007, 2009) and fuzzy counterparts defined in this article refer to the whole family. The fuzzy index is aimed at removing dual cutoff present in Alkire Foster methodology. The first cutoff was not problematic, we simply treated normalized gap function as a natural fuzzy membership function as it was already incorporated in indices belonging to this specific family. All research effort focused on removing the second, cross-dimensional cutoff. We proposed a function which in addition to individuals deprived in more than k dimension takes into account – with lower weight – those deprived in less than k. The index fulfills bunch of desired properties. The methodology developed in this paper is now being applied to *Badanie Budżetów Gospodarstw Domowych* (Household Budget Survey) and *Badanie Spójności Społecznej* (Social Cohesion Study) to evaluate the extent of multidimensional poverty in Polish regions.

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ROZMYTY WSKAŹNIK UBÓSTWA WIELOWYMIAROWEGO

STRESZCZENIE

Wskaźnik ubóstwa wielowymiarowego (MPI) to nowa miara ubóstwa, z której korzysta Organizacja Narodów Zjednoczonych ds. Rozwoju. Konstrukcja MPI jest odpowiedzią na wezwanie do przyjęcia wielowymiarowej wizji ubóstwa (Sen, Anand, 1997). Miara ta pozwala ocenić ubóstwo na poziomie indywidualnym. Biedna osoba to taka, która jest wykluczona w kilku wymiarach. Miara ta jednocześnie pozwala ocenić rozmiary ubóstwa. Metodologia MPI korzysta z tzw. podwójnego odcięcia w celu sklasyfikowania danej osoby jako ubogiej (lub nie). To znaczy, pierwsza linia ubóstwa jest wyznaczona dla każdego wymiaru oddzielnie, a druga linia to minimalna liczba wymiarów, którą uznaje się za konieczną do tego, aby uznać osobę za ubogą. Taka arbitralność w wyznaczaniu granicy ubóstwa powoduje, że naturalne staje się rozszerzenie MPI o zbiory rozmyte. Rozmyte podejście do pomiaru ubóstwa odzwierciedla niepewność, która pojawia się w pomiarze zjawisk społecznych. Większość zmiennych używanych w bada-

niach społecznych jest jakościowa, tj. ma postać uporządkowanych opisowych kategorii. Jakakolwiek ostra granica co do tego, gdzie zaczyna się analfabetyzm lub niedożywienie, jest uznaniowa. Zamiast tego w podejściu rozmytym członkostwo w ubóstwie nie jest 0–1, ale ma różne stopnie: od niskiego do wysokiego. W tym artykule rozwinięto miary, które są rozmytym uogólnieniem indeksu MPI oraz udowodniono ich własności.

Słowa kluczowe: wielowymiarowe ubóstwo, współczynnik ubóstwa, zbiory rozmyte.

Klasyfikacja JEL: I3, I32, D63